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Rigidity of Teichmüller Space

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Abstract

This is the third paper in the series ([DM1] and [DM2] being the previous two) where we study harmonic maps into the Weil-Petersson completion $\overline{\mathcal{T}}$ of Teichmüller space. In this paper, we prove that the singular set of a harmonic map from a smooth n-dimensional Riemannian domain to $\overline{\mathcal{T}}$ has Hausdorff dimension at most $n-2$, and moreover, the harmonic map has certain decay near the singular set. Combined with the earlier work of Schumacher and Jost-Yau, this implies the holomorphic rigidity of Teichmüller space. In addition, our results provide a harmonic maps proof of both the high rank and the rank one superrigidity of the mapping class group proved via other methods by Farb-Masur and Yeung.

1 Introduction

Let $\mathcal{T}$ denote the Teichmüller space of an oriented surface S of genus g and p marked points such that $k = 3g-3+p > 0$. The Teichmüller space endowed with the Weil-Petersson metric is an incomplete Riemannian manifold (cf. [Ch] and [Wo2]). Its metric completion $\overline{\mathcal{T}}$ is an NPC (non positively curved) space (cf. [Yam]), and its boundary $\partial\mathcal{T}$ can be stratified by lower dimensional Teichmüller spaces corresponding to nodal surfaces formed by pinching a finite set of nontrivial, nonperipheral, simple closed curves (cf. [Wo1]). Thus,

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$\overline{\mathcal{T}}$ is a stratified space (with the original Teichmüller space $\mathcal{T}$ being the top dimensional stratum). In this paper, we study harmonic maps $u : \Omega \to \overline{\mathcal{T}}$ from a Riemannian domain. Given such a map, we define the regular set $\mathcal{R}(u)$ to be the set of points in Ω that possess a neighborhood mapping into a single stratum and the singular set $\mathcal{S}(u)$ to be its complement. The main results of this paper are:

Theorem 1 *Let $\mathcal{T} = \mathcal{T}(S)$ be Teichmüller space of an oriented surface S of genus g and p marked points such that $k = 3g - 3 + p > 0$ and $\overline{\mathcal{T}}$ be its metric completion with respect to the Weil-Petersson metric. If $u : \Omega \to \overline{\mathcal{T}}$ is a harmonic map from an n-dimensional Lipschitz Riemannian domain, then*

$$\dim_{\mathcal{H}} \Big(\mathcal{S}(u)\Big) \leq n - 2.$$

Theorem 2 *Let $u : \Omega \to \overline{\mathcal{T}}$ be as in Theorem 1. For any compact subdomain Ω_1 of Ω, there exists a sequence of smooth functions $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $\mathcal{S}(u) \cap \overline{\Omega_1}$, $0 \leq \psi_i \leq 1$ and $\psi_i(x) \to 1$ for all $x \in \Omega_1 \backslash \mathcal{S}(u)$ such that*

$$\lim_{i \to \infty} \int_\Omega |\nabla\nabla u||\nabla\psi_i| \; d\mu = 0.$$

The theory of harmonic maps to singular spaces was initiated by Gromov and Schoen in the seminal paper [GS], where they mainly consider harmonic maps to Euclidean buildings. This work was subsequently extended for harmonic maps to maps into more general NPC spaces by Korevaar-Schoen and Jost (cf. [KS1], [KS2] and [Jo]). In [DM1], we developped techniques to study harmonic maps into spaces with an *asymptotic product structure*. By this, we mean NPC spaces that satisfy the Assumptions of Section 2.5 (see below). Euclidean buildings and differentiable manifold complexes are examples of such spaces.

The Weil-Petersson completion of Teichmüller space also fits into this framework. Indeed, according to [Yam], [DW], [Wo1] and [DM3], the Weil-Petersson completion of a Teichmüller space near a boundary point is asymptotically isometric to the product of a boundary stratum $\mathcal{T}'$ and a normal space $\overline{\mathbf{H}}^{k-j} = \overline{\mathbf{H}} \times \ldots \times \overline{\mathbf{H}}$, where we call $\overline{\mathbf{H}}$ the *model space* (see Section 2.2 for a precise definition of $\overline{\mathbf{H}} = \mathbf{H} \cup \{P_0\}$ given as a metric completion of the incomplete Riemannian surface $\mathbf{H}$). Since the boundary stratum $\mathcal{T}'$ is a lower dimensional Teichmüller space which is a smooth Hermitian manifold,

the singular behavior of the Weil-Petersson geometry is completely captured by the model space $\overline{\mathbf{H}}$. For one, the Gauss curvature of $\mathbf{H}$ approaches $-\infty$ near its boundary reflecting the sectional curvature blow up of $\mathcal{T}$ near $\partial\mathcal{T}$. Moreover, the non-local compactness of $\overline{\mathcal{T}}$ is also captured by $\overline{\mathbf{H}}$. Indeed, a geodesic ball in $\overline{\mathbf{H}}$ centered at a boundary point is not compact. This lack of compactness *imposes severe challenges in the theory of harmonic maps* which have not been addressed before, and the core of this paper is to deal with these phenomena.

The above regularity theorems have important consequences in proving rigidity of Teichmüller space and the mapping class group. The first is

Theorem 3 (Holomorphic rigidity of Teichmüller space) *Let $\Gamma = Mod(S)$ denote the mapping class group of an oriented surface S of genus g and p marked points such that $k = 3g - 3 + p > 0$. If Γ acts on a contractible Kähler manifold $\tilde{M}$ such that*

- *(i) there is a finite index subgroup Γ' of Γ such that $M := \tilde{M}/\Gamma'$ is a smooth manifold,*
- *(ii) M is a quasiprojective variety and*
- *(iii) M admits a compactification $\overline{M}$ such that the codimension of $\overline{M}\backslash M$ is at least 3,*

then $\tilde{M}$ is equivariantly biholomorphic or conjugate biholomorphic to the Teichmüller space $\mathcal{T} = \mathcal{T}(S)$.

The history behind the problem of holomorphic rigidity of Kähler manifolds starts in 1960 with the work of Calabi and Vesentini [CV] where they showed that compact quotients of bounded symmetric domains of dimension at least 2 do not admit any nontrivial infinitesimal holomorphic deformation. The celebrated Mostow rigidity theorem of 1968 implies that two compact quotients of the ball of complex dimension at least 2 with isomorphic fundamental groups are isometric and thus biholomorphic or conjugate biholomorphic. Yau conjectured that the same holds for any two compact Kähler manifolds of dimension at least 2 and negative sectional curvature. This conjecture was subsequently proved in 1980 using harmonic maps by

Siu [Siu] in the case when one of the manifolds has strong negative curvature. Furthermore, Jost and Yau [JoYa] in 1987 extended the above results to symmetric domains that are not necessarily compact, provided that the group acts with orbifold singularities and the quotient has a reasonable compactification. Given that in many aspects Teichmüller space resembles a complex symmetric domain, Jost and Yau also stated the holomorphic rigidity theorem for Teichmüller space (cf. [JoYa]). As proved by Schumacher [Sch], the Teichmüller space with the Weil-Petersson metric has strong negative curvature, thus [Sch] and [JoYa] show that holomorphic rigidity of Teichmüller space holds, *provided that there exists an appropriate harmonic map whose image is contained in the interior of Teichmüller space.*

On the other hand, the incompleteness of $\mathcal{T}$ with respect to the Weil-Petersson metric mentioned at the beginning of this section *poses a great difficulty in finding such a harmonic map*. Thus, we need to study the behavior of harmonic maps at the singular points, and this is the content of the regularity theorems above. By considering the metric completion $\overline{\mathcal{T}}$ of $\mathcal{T}$ with respect to the Weil-Petersson metric, the space $\overline{\mathcal{T}}$ bears resemblance to a Hadamard manifold; in particular, the existence of an equivariant harmonic map into $\overline{\mathcal{T}}$ under some nondegeneracy conditions was shown in [DW]. On the other hand, due to the singular nature of $\overline{\mathcal{T}}$, the harmonic map $u : \tilde{M} \to \overline{\mathcal{T}}$ may exhibit singular behavior. However, Theorem 1 and Theorem 2 assert that the singular set $\mathcal{S}(u)$ of u is of Hausdorff codimension at least 2 and moreover u has certain decay near the singular set. This is enough to enable us to apply the Bochner formula and prove rigidity.

We now describe the organization of the paper and explain the main ideas. In Section 2.1, we give the basic definitions of the *order function*, *monotonicity* and *blow-up maps* for harmonic maps to NPC spaces. For convenience, we state the above for a more general class of maps that include asymptotically harmonic maps. In Section 2.2, we describe limits of harmonic maps into the model space $\mathbf{H}$ which, as explained above, models the behavior of harmonic maps near $\partial\mathcal{T}$. In Section 2.3, we discuss different coordinate systems on the model space $\mathbf{H}$. We also review one of the main Theorems from [DM2] (cf. Theorem 16), which asserts that certain subsets of $\overline{\mathbf{H}}$ near the boundary point satisfy a property analogous to the *essentially regular* property in [GS]. This is the first crucial ingredient in the proof of our regularity theorem. Additionally, we also prove the crucial Lemma 18, which plays the role of the *effectively contained* property in [GS]. In Section 2.4 and

Section 2.5, we define the structure of the singular set of u and state the basic assumptions needed in order to apply [DM1]. *For asymptotic harmonic maps, these assumptions will be verified in Section 5.* We first state Assumptions 1-4 which imply the *target variation formula* (cf. Theorem 19) and the crucial subharmonicity statement (cf. Lemma 20). We then state the rest of the Assumptions 5-6 and Theorem 21 which asserts that the singular component v of harmonic map has a well defined order. In particular, we can define blow-up maps and deduce that the singular set of points of order greater than one is of Hausdorff codimension at least 2.

The heart of our argument lies in Section 3. This is an adaptation to our situation of [GS] Theorem 5.1. We have separated our argument in two parts. The first is the Inductive Lemma 24. The second is its consequences; in particular, Proposition 25 implies that approximately harmonic maps cannot hit the boundary near a point of order one and Corollary 28 implies that the singular set of a harmonic map into $\overline{\mathbf{H}}$ is of codimension at least 2.

Section 4 contains certain technical results needed in later sections. In particular, we prove that given a harmonic map into $\overline{\mathcal{T}}$, the set of points of order greater than one has Hausdorff codimension at least 2 (cf. Proposition 31). In Section 5, we verify that Assumptions 1-6 from [DM1] hold for harmonic maps into $\overline{\mathcal{T}}$, and this completes the proof of our Theorem 1 and Theorem 2. In Section 6, we specialize the the case when the domain dimension is 2. We prove that there are no singular point in this case (cf. Theorem 5 below).

Section 7 contains the main applications of Theorem 1 and Theorem 2; most importantly, we give the proof of the holomorphic rigidity Theorem 3. Additionally, as a by product, we provide a harmonic maps proof of the following theorem due to Farb-Masur and Yeung.

Theorem 4 (Superrigidity of the MCG, cf. [FaMa], [Ye]) *Let $\tilde{M} = G/K$ be an irreducible symmetric space of noncompact type, other than $SO_0(p,1)/SO(p) \times SO(1)$, $SU_0(p,1)/S(U(p) \times U(1))$. Let Λ be a discrete subgroup of G with finite volume quotient and let $\rho : \Lambda \to Mod(S)$ be sufficiently large, where $Mod(S)$ denotes the mapping class group of an oriented surface S of genus g and p marked points such that $k = 3g-3+p > 0$. If the rank of $\tilde{M}$ is ≥ 2, we assume additionally that Λ is cocompact. Then any homomorphism ρ of Λ into the mapping class group $Mod(S)$ has finite image.*

As a final comment, we would like to point out that Theorem 1 only implies that the singular set $\mathcal{S}(u)$ of u is of codimension at least 2 (or more precisely that u maps a connected domain into a single stratum up to codimension at 2) and does not necessarily imply that u maps into the interior of $\mathcal{T}$ as originally asserted in [JoYa]. Of course, a posteriori, the Bochner formula implies that the Jost-Yau claim indeed holds for the cases of interest in this paper (see Section 7). For two dimensional domains, this assertion is generally true; namely,

Theorem 5 *If $u : \Omega \to \overline{\mathcal{T}}$ is a harmonic map from an connected Lipschitz domain Ω in a Riemann surface, then there exists a single stratum $\mathcal{T}'$ of $\overline{\mathcal{T}}$ such that $u(\Omega) \subset \mathcal{T}'$.*

It is reasonable to conjecture that this assertion holds for higher dimensions; however, this not needed for the applications discussed in this article.

Conjecture 6 *If $u : \Omega \to \overline{\mathcal{T}}$ is a harmonic map from a connected n-dimensional Lipschitz Riemannian domain, then there exists a single stratum $\mathcal{T}'$ of $\overline{\mathcal{T}}$ such that $u(\Omega) \subset \mathcal{T}'$.*

Acknowledgement. In the special case when the domain Ω is a region in a Rieman surface, it was first proved by Wentworth that the singular set $\mathcal{S}(u)$ is empty for a harmonic map $u : \Omega \to \overline{\mathbf{H}}$ (cf. [W]). His method is strictly *two dimensional* using the Hopf differential associated with the harmonic map, thus cannot be generalized to arbitrary dimensions. Even though our method is completely different from his, some of the preliminary results used in this paper, for example the structure of limits of harmonic maps to $\overline{\mathbf{H}}$, have their origin in [W]. We would like to thank Richard Wentworth for sharing his unpublished manuscript with us. Additionally, we would like to thank Bill Minicozzi for his continuous support of this project, Bernie Shiffman for the reference [Schi] and R. Schoen, K. Uhlenbeck, S Wolpert and S. T. Yau for sharing their insights on the subject with us.

2 Preliminaries

2.1 Harmonic maps into NPC spaces

We first recall some preliminary facts regarding harmonic maps into NPC spaces and some related concepts. Let Ω be a n-dimensional Lipschitz Rie-

mannian domain with metric g and (Y,d) an NPC space. For a finite energy map $u:\Omega \to (Y,d)$, let $|\nabla u|^2$ denote the energy density as defined in [KS1] (1.10v). A map u is said to be *harmonic* if it is energy minimizing amongst all finite energy maps with the same boundary value on every bounded Lipschitz subdomain $\Omega' \subset \Omega$ (cf. [KS1]). A harmonic map is locally Lipschitz by [KS1] Theorem 2.4.6.

Let $v:\Omega \to (Y,d)$ be a map (not necessarily harmonic). For $x_0 \in \Omega$, define

$$E^v_{x_0}(\sigma) := \int_{B_\sigma(x_0)} |\nabla v|^2 d\mu \ \text{ and } \ I^v_{x_0}(\sigma) := \int_{\partial B_\sigma(x_0)} d^2(v, v(x_0)) d\Sigma.$$

In the sequel, we will often suppress the subscript x_0 if the choice of the point x_0 is clear from the context. The *order of the map* v *at* x_0 is

$$Ord^v(x_0) := \lim_{\sigma \to 0} \frac{\sigma \ E^v(\sigma)}{I^v(\sigma)} \text{ if this limit exists.}$$

The set

$$\mathcal{S}_0(v) := \{x_0 \in \Omega : \ Ord^v(x_0) \text{ exists and is } > 1\}$$

is the *higher order points* of v.

Remark 7 If $u:\Omega \to (Y,d)$ is a harmonic map and $x_0 \in \Omega$, there exists a constant $c>0$ depending only on the C^2 norm of the domain metric (with $c=0$ when Ω is a Euclidean metric) such that

$$\sigma \in (0,\sigma_0) \mapsto e^{c\sigma^2} \frac{\sigma \ E^u_{x_0}(\sigma)}{I^u_{x_0}(\sigma)} \text{ is non-decreasing} \tag{1}$$

for any $x_0 \in \Omega$ and $\sigma_0 > 0$ sufficiently small. Thus, $Ord^u(x_0)$ exists for all $x_0 \in \Omega$. Furthermore, we have that $Ord^u(x_0) \geq 1$,

$$\sigma \mapsto e^{c\sigma^2} \frac{E^u_{x_0}(\sigma)}{\sigma^{n-2+2\alpha}} \text{ and } \sigma \mapsto e^{c\sigma^2} \frac{I^u_{x_0}(\sigma)}{\sigma^{n-1+2\alpha}} \text{ are non-decreasing.} \tag{2}$$

These monotonicity statements above follow from Section 1.2 of [GS] combined with [KS1], [KS2] to justify various technical steps.

We now define the notion of *blow up maps* of a map $v:\Omega \to (Y,d)$ (not necessarily harmonic) at $x_0 \in \Omega$. Below, $B_R(0) \subset \mathbf{R}^n$ and g_0 is the standard

Euclidean metric. We identify $x_0 = 0$ and consider $v : (B_R(0), g_0) \to (Y, d)$ by assuming that the standard Euclidean coordinates $(x^1, \dots, x^n)$ are normal coordinates centered at x_0 with respect to the domain metric g. Such metric we will henceforth call *normalized*. For $\sigma_0 > 0$ small and a function

$$\nu : (0, \sigma_0) \to \mathbf{R}_{>0} \quad \text{with} \quad \lim_{\sigma \to 0^+} \nu(\sigma) = 0,$$

the *blow up map* of v at $x_0 = 0$ with *scaling factor* $\nu(\sigma)$ for $\sigma \in (0, \sigma_0)$ is the map defined by

$$v_\sigma : (B_1(0), g_\sigma) \to (Y, d_\sigma), \quad v_\sigma(x) = v(\sigma x) \tag{3}$$

with

$$g_\sigma(x) = g(\sigma x) \quad \text{and} \quad d_\sigma(P, Q) = \nu(\sigma)^{-1} d(P, Q).$$

The *approximating harmonic map* is the harmonic map

$$w_\sigma : (B_1(0), g_\sigma) \to (Y, \nu(\sigma)^{-1} d) \quad \text{with} \quad w_\sigma|_{\partial B_1(0)} = v_\sigma|_{\partial B_1(0)}.$$

Remark 8 If $u : \Omega \to (Y, d)$ is a harmonic map and $x_0 \in \Omega$, then the blow up map

$$u_\sigma : (B_1(0), g_\sigma) \to (Y, d_\sigma) \ \text{ with blow up factor } \ \nu(\sigma) = \sqrt{\frac{I^u(\sigma)}{\sigma^{n-1}}}$$

at x_0 is a harmonic map (hence $u_\sigma = w_\sigma$) and $\{u_\sigma\}$ has uniformly bounded energy. For any sequence $\sigma_i \to 0$, we can find a subsequence $\sigma_{i'} \to 0$ such that $\{u_{\sigma_{i'}}\}$ converges locally uniformly in the pullback sense (cf. [KS2] Definition 3.3) to a homogeneous, degree $\alpha = Ord^u(0)$ harmonic map $u_* : (B_1(0), g_0) \to (Y_0, g_0)$ from a Euclidean ball into a NPC space. Here, homogeneous implies to the property that for any $\xi \in \partial B_1(0)$, the image $\{u_*(t\xi) : \ t \in (0,1)\}$ is a geodesic and

$$d(u_*(t\xi), u_*(0)) = t^\alpha d(u_*(\xi), u_*(0)), \quad \forall t \in (0, 1).$$

For more details, see [GS] with [KS1], [KS2] to justify various technical steps.

The following definition and theorem are from [DM1] Appendix 2.

Definition 9 Let $v : \Omega \to (Y, d)$ be a finite energy continuous map from a Lipschitz Riemannian domain into an NPC space and let $\mathcal{S}$ be a closed subset of Ω. We say v *satisfies (P1) and (P2) with respect to* $\mathcal{S}$ if it satisfies the properties below.

(P1) At any $x_0 \in \mathcal{S}$, we require that v has a well defined order at x_0 in the sense that it satisfies the following property: Assume that v is not constant in any neighborhood of x_0 and that there exist constants $c > 0$ and $R_0 > 0$ such that for any $x_0 \in \mathcal{S}$,

$$Ord^v(x_0) := \lim_{\sigma \to 0} \frac{\sigma E^v_{x_0}(\sigma)}{I^v_{x_0}(\sigma)} \text{ exists}$$

and

$$Ord^v(x_0) \leq e^{c\sigma} \frac{\sigma E^v_{x_0}(\sigma)}{I^v_{x_0}(\sigma)}, \quad \forall \sigma \in (0, R_0).$$

(P2) Given a sequence $\sigma_i \to 0$, there exists a subsequence (which we call again σ_i by a slight abuse of notation) such that the blow up maps $\{v_{\sigma_i}\}$ and the approximating harmonic maps $\{w_{\sigma_i}\}$ converge locally uniformly in the pullback sense to a homogeneous harmonic map $v_0 : (B_1(0), \delta) \to (Y_0, d_0)$ for some NPC space. For any $r \in (0, 1)$,

$$\lim_{i \to \infty} \sup_{B_r(0)} d(v_{\sigma_i}, w_{\sigma_i}) = 0.$$

Furthermore, for σ_i sufficiently small and any sequence $\{x_i\} \subset \sigma_i^{-1}\mathcal{S} \cap B_{\frac{1}{2}}(0)$, $R \in (0, \frac{1}{4})$, there exists $\{r_i\} \subset (\frac{R}{2}, R)$ such that

$$\lim_{i \to \infty} \left| E^{v_{\sigma_i}}_{x_i}(r_i) - E^{w_{\sigma_i}}_{x_i}(r_i) \right| = 0.$$

Note that by [DM1] formula (173),

$$\lim_{i \to \infty} \frac{r\sigma_i E^v_{x_0}(r\sigma_i)}{I^v_{x_0}(r\sigma_i)} = \lim_{i \to \infty} \frac{r E^{v_{\sigma_i}}_{x_0}(r)}{I^{v_{\sigma_i}}_{x_0}(r)} = \frac{r E^{v_0}_{x_0}(r)}{I^{v_0}_{x_0}(r)}, \quad r \in (0, R].$$

Hence, by the homogeneity of v_0,

$$Ord^v(x_0) = Ord^{v_0}(x_0). \tag{4}$$

Theorem 10 ([DM1] Appendix 2) *Let $v : \Omega \to (Y, d)$ be a map satisfying properties (P1) and (P2) with respect to $\mathcal{S} \subset \Omega$. If there exists $\epsilon_0 > 0$ such that for every $x_0 \in \mathcal{S}$, the homogeneous harmonic map v_0 of $(P1)$ associated with the point x_0 has the property that either*

$$Ord^{v_0}(0) = 1 \quad or \quad Ord^{v_0}(0) \geq 1 + \epsilon_0$$

and

$$\dim(\mathcal{S}_0(v_0)) \leq n - 2,$$

then the set of higher order points in $\mathcal{S}$ is of codimension at least 2; i.e.

$$\dim(\mathcal{S}_0(v) \cap S) \leq n - 2.$$

Corollary 11 *Let $u : (\Omega, g) \to (Y, d)$ be a harmonic map from a Riemannian domain into an NPC space. If, at every point $x_0 \in \Omega$, a tangent map u_* of u at x_0 has the property that for either $Ord^{u_*}(0) = 1$ or $Ord^{u_*}(0) \geq 1 + \epsilon_0$ and $\dim(\mathcal{S}_0(u_*)) \leq n - 2$, then $\dim(\mathcal{S}_0(u)) \leq n - 2$.*

PROOF. Combine Remark 8 and Theorem 10. Q.E.D.

2.2 Harmonic maps into the model space $\overline{\mathbf{H}}$

The boundary $\partial\mathcal{T}$ of the Weil-Petersson completion $\overline{\mathcal{T}}$ of Teichmüller space $\mathcal{T}$ is stratified by lower dimensional Teichmüller spaces and the normal space to each stratum is a product of copies of a singular space $\overline{\mathbf{H}}$ called the model space. The significance of $\overline{\mathbf{H}}$ is that it captures the singular behavior of the Weil-Petersson geometry of $\overline{\mathcal{T}}$. To define it, first consider the smooth Riemann surface $(\mathbf{H}, g_{\mathbf{H}})$ where

$$\mathbf{H} = \{(\rho, \phi) \in \mathbf{R}^2 : \rho > 0, \phi \in \mathbf{R}\} \ \text{ and } \ g_{\mathbf{H}} = d\rho^2 + \rho^6 d\phi^2.$$

It is straightforward to check that $(\mathbf{H}, g_{\mathbf{H}})$ is a geodesically convex Riemann surface of Gauss curvature $K = -\frac{6}{\rho^2}$. Let $d_{\mathbf{H}}$ denote the distance function on $\mathbf{H}$ induced by $g_{\mathbf{H}}$. The *model space* is the metric completion of $\mathbf{H}$ with respect to $d_{\mathbf{H}}$, the distance function induced by $g_{\mathbf{H}}$. More precisely, we let

$$\overline{\mathbf{H}} = \mathbf{H} \cup \{P_0\}$$

where P_0 is the axis $\rho = 0$ identified to a single point and the distance function $d_{\mathbf{H}}$ on $\mathbf{H}$ is extended to $\overline{\mathbf{H}}$ by setting $d_{\mathbf{H}}(Q, P_0) = \rho$ for $Q = (\rho, \phi) \in \mathbf{H}$. Then $\overline{\mathbf{H}}$ with the distance function $d_{\mathbf{H}}$ is an NPC space. The distance function on the product space $\overline{\mathbf{H}}^{k-j} = \overline{\mathbf{H}} \times \ldots \times \overline{\mathbf{H}}$ will be denoted simply by $d(\cdot, \cdot)$.

The following was proved in the appendix of [DM2].

Lemma 12 *There exists $\epsilon_0 > 0$ such that if a sequence of harmonic maps $\{w_i = (w_i^\rho, w_i^\phi) : (B_1(0), g_i) \to \overline{\mathbf{H}}\}$ with uniformly bounded energy converges locally uniformly in the pullback sense to a homogenous harmonic map $v_0 : (B_1(0), g_0) \to (Y_0, d_0)$ into an NPC space, $\lim_{i\to\infty} w_i(0) = P_0$ and g_i converges in C^k $(k = 0, 1, \ldots)$ to the Euclidean metric g_0, then*

$$Ord^{v_0}(0) = 1 \quad or \quad Ord^{v_0}(0) \geq 1 + \epsilon_0.$$

and

$$\dim_{\mathcal{H}}(\mathcal{S}_0(v_0)) \leq n - 2.$$

Moreover, if $Ord^{v_0}(0) = 1$, then the pullback distance function of v_0 is equal to that of a linear function.

2.3 Coordinates of the model space H

In this section, we introduce two global coordinates on $\mathbf{H}$ different from the (ρ, ϕ) coordinates discussed in Section 2.2. We will refer to (ρ, ϕ) as the *original coordinates.*

2.3.1 The homogeneous coordinates (ρ, Φ)

For the *homogeneous coordinates* (ρ, Φ) of $\mathbf{H}$, the first coordinate function ρ is the same as that of the original coordinates, but the second coordinate function Φ is defined by setting

$$\Phi = \rho^3 \phi.$$

The term *homogeneous* refers to that fact that the metric $g_{\mathbf{H}}$ is invariant under the scaling

$$\rho \to \lambda\rho, \;\; \Phi \to \lambda\Phi.$$

This can be checked by a straightforward computation (cf. [DM2] or [W]). Thus, if we define

$$\lambda P = \begin{cases} (\lambda\rho, \lambda\Phi) & \text{if } P = (\rho, \Phi) \neq P_0 \\ P_0 & \text{if } P = P_0 \end{cases} \tag{5}$$

then the distance function is homogeneous of degree 1 with respect to the map $P \mapsto \lambda P$; in other words

$$d(\lambda P, \lambda Q) = \lambda d(P, Q), \quad \forall P, Q \in \overline{\mathbf{H}},\ \lambda \in (0, \infty). \tag{6}$$

The importance of the homogeneous coordinates is that they can be used to view blow up maps $\{u_\sigma\}$ (cf. (3)) of a map $v : (B_R(0), g) \to \overline{\mathbf{H}}$ as again maps into $\overline{\mathbf{H}}$. Indeed, write

$$v = (v_\rho, v_\Phi)$$

in the homogeneous coordinates (ρ, Φ) and let $\sigma > 0$ small. By (6), we can think of the blow up maps as

$$v_\sigma = (v_{\sigma\rho}, v_{\sigma\Phi}) : (B_1(0), g_\sigma) \to \overline{\mathbf{H}} \tag{7}$$

given by

$$v_{\sigma\rho}(x) = \nu^{-1}(\sigma) v_\rho(\sigma x) \ \text{ and } \ v_{\sigma\Phi}(x) = \nu^{-1}(\sigma) v_\Phi(\sigma x).$$

2.3.2 The coordinates (ϱ, φ) via symmetric geodesics

In [DM2], we introduced another global coordinate system of $\mathbf{H}$ constructed by foliating $\mathbf{H}$ by geodesics. In order to define these coordinates, we first need the notion of a *symmetric geodesic* from [DM2]. This is defined to be an arclength parameterized geodesic $\gamma : (-\infty, \infty) \to \mathbf{H}$ such that if we write $\gamma = (\gamma_\rho, \gamma_\phi)$ with respect to the original coordinates (ρ, ϕ) of $\mathbf{H}$, then

$$\gamma_\rho(s) = \gamma_\rho(-s) \ \text{ and } \ \gamma_\phi(s) = -\gamma_\phi(-s).$$

A map $l = (l_\rho, l_\phi) : B_1(0) \to \mathbf{H}$ is said to be a *symmetric homogeneous degree 1 map* if

$$l(x) = \gamma(Ax^1) \tag{8}$$

for some $A > 0$ and a symmetric geodesic γ. We call the number A the *stretch* of l. Furthermore, a *translation isometry* is an isometry

$$T : \overline{\mathbf{H}} \to \overline{\mathbf{H}}$$

defined by setting

$$T(P_0) = P_0 \text{ and } T(\rho, \phi) = (\rho, \phi + c)$$

for some $c \in \mathbf{R}$.

The following lemma, proved in [DM2], explains why symmetric homogeneous degree 1 maps naturally arise in the study of harmonic maps into $\overline{\mathbf{H}}$.

Lemma 13 *Assume $\{w_i : (B_\rho(0), g_i) \to \overline{\mathbf{H}}\}$ is a sequence of harmonic maps with uniformly bounded total energy converging locally uniformly in the pull-back sense to a homogeneous harmonic map $v_0 : (B_\rho(0), g_0) \to (Y_0, d_0)$ into an NPC space, $\lim_{i\to\infty} w_i(0) = P_0$ and g_i converges in C^k $(k = 0, 1, \dots)$ to the Euclidean metric g_0. If $Ord^{v_0}(0) = 1$, then there exists $A > 0$, a rotation $R : \mathbf{R}^n \to \mathbf{R}^n$, a sequence of translation isometries $T_i : \overline{\mathbf{H}} \to \overline{\mathbf{H}}$, a sequence of symmetric homogeneous degree 1 maps $l_i : B_\rho(0) \to \overline{\mathbf{H}}$ with $d_{\mathbf{H}}(P_0, l_i(0)) \to 0$ and stretch $\to A$ such that*

$$\lim_{i\to\infty} \sup_{B_r(0)} d_{\mathbf{H}}(w_i, T_i \circ l_i \circ R) = 0, \ \forall r \in (0, \rho).$$

We now consider one parameter families of geodesics

$$c = (c_\rho, c_\phi) : (-\infty, \infty) \times (-\infty, \frac{3}{2}) \to \mathbf{H} \tag{9}$$

satisfying the following:

- $t \mapsto c_\rho(0, t)$ satisfies the equation $\dfrac{\partial c_\rho}{\partial t}(0, t) = c_\rho^3(0, t),$ (10)
- $c_\rho(0, 1) = 1$ and $c_\phi(0, t) = 0$ for all $t \in (-\infty, \frac{3}{2}),$ (11)
- $s \mapsto c^t(s) = c(s, t)$ is a unit speed symmetric geodesic. (12)

The parameters s and t define coordinates of $\mathbf{H}$ via the map

$$(s, t) \mapsto c(s, t).$$

Given a symmetric homogeneous degree 1 map $l : B_1(0) \to \mathbf{H}$, let

$$l(x) = (l_s(x), l_t(x))$$

be the expression of l with respect to the coordinates (s,t). By the construction of the coordinates (s,t),

$$\exists t_* \in (-\infty, \frac{3}{2}) \text{ such that } l(x) = (l_s(x), l_t(x)) = (Ax^1, t_*) \tag{13}$$

where A is the stretch of l. We refer to the number t_* as the *address* of l. In particular, we have

$$d_{\mathbf{H}}(P_0, l(0)) = c_\rho(0, t_*) \tag{14}$$

We define another set of coordinates (ϱ, φ), by applying a linear change of variables

$$(s,t) \mapsto (\varrho, \varphi) = (s, t - t_*). \tag{15}$$

We note that the definition (ϱ, φ) depends on t_* and we will say that (ϱ, φ) is *anchored* at t_* By definition, the map l in (13) is given in coordinates (ϱ, φ) by

$$l(x) = (l_\varrho(x), l_\varphi(x)) = (Ax^1, 0). \tag{16}$$

We also record the following simple lemma (cf. [DM2]).

Lemma 14 *If $P_1, P_2 \in \mathbf{H}$ are given in coordinates (ϱ, φ) as $P_1 = (\varrho_1, \varphi_1)$ and $P_2 = (\varrho_2, \varphi_2)$, then*

$$|\varrho_1 - \varrho_2| \leq d_{\mathbf{H}}(P_1, P_2).$$

We write the metric $g_{\mathbf{H}}$ with respect to coordinates (ϱ, φ) as

$$g_{\mathbf{H}} = d\varrho^2 + \mathcal{J}(\varrho, \varphi) d\varphi^2 \tag{17}$$

As seen in [DM2], this local expression of $g_{\mathbf{H}}$ with respect to (ϱ, φ) is close to the local expression $g_{\mathbf{H}} = d\rho^2 + \rho^6 d\phi^2$ with respect to (ρ, ϕ). In particular, there exists a constant $C > 0$ such that

$$\varrho^3 \leq \mathcal{J}(\varrho, \varphi) \leq C(\varrho + c_\rho(0, \varphi + t_*))^3. \tag{18}$$

For $\varphi_0 > 0$, define the subset

$$\overline{\mathbf{H}}[\varphi_0, t_*] := \{(\varrho, \varphi) \in \mathbf{H} : |\varphi| \leq \varphi_0\}. \tag{19}$$

Since the level sets $\varphi = \varphi_0$ and $\varphi = -\varphi_0$ are images of geodesic lines $\varrho \mapsto c(\varrho, \varphi_0 + t_*)$ and $\varrho \mapsto c(\varrho, -\varphi_0 + t_*)$, the subset $\mathbf{H}[\varphi_0, t_*]$ is totally geodesic. We also define

$$a[\varphi_0, t_*] := c_\rho(0, \varphi_0 + t_*) = \max_{\{\varphi : |\varphi| \leq \varphi_0\}} c_\rho(0, \varphi + t_*). \tag{20}$$

In particular, the component function $\mathcal{J}(\varrho, \varphi)$ of $g_{\mathbf{H}}$ is bounds of the form

$$\varrho^3 \leq \mathcal{J}(\varrho, \varphi) \leq C(\varrho + a[\varphi_0, t_*])^3 \text{ for } (\varrho, \varphi) \in \overline{\mathbf{H}}[\varphi_0, t_*].$$

In [DM2], we showed that harmonic maps into $\overline{\mathbf{H}}[\phi_0, t_*]$ are close to being affine; indeed, Theorem 16 below is the main result of that paper. In particular, $\overline{\mathbf{H}}[\varphi_0, t_*]$ is close to being *essentially regular* in the sense of Gromov-Schoen [GS]. We refer to the introduction of [DM2] for a detailed explanation of this notion. First, we need the following definition.

Definition 15 We say that a map $l = (l_\varrho, l_\varphi) : B_1(0) \to \overline{\mathbf{H}}$ written with respect to coordinates (ϱ, φ) is an *almost affine map* if $l_\varrho(x) = \vec{a} \cdot x + b$ for $\vec{a} \in \mathbf{R}^n$ and $b \in \mathbf{R}$, i.e. l_ϱ is an affine function.

Theorem 16 (cf. [DM2]) *Let $R \in [\frac{1}{2}, 1)$, $E_0 > 0$, $A_0 > 0$ and a normalized metric g on $B_R(0)$ be given. Then there exist $C \geq 1$ and $\alpha > 0$ depending only on E_0, A_0 and g with the following property:*

For $\varphi_0 > 0$ and $\vartheta \in (0, 1]$, if $\mathbf{B}_{A_0\vartheta}(P_0)$ is a geodesic ball of radius $A_0\vartheta$ centered at P_0 in $\overline{\mathbf{H}}$, if

$$w : (B_{\vartheta R}(0), g_s) \to \overline{\mathbf{H}}[\frac{\varphi_0}{\vartheta^2}, t_*] \cap \mathbf{B}_{A_0\vartheta}(P_0)$$

is a harmonic map with

$$a[\frac{\varphi_0}{\vartheta^2}, t_*] \leq \frac{\vartheta}{2} \tag{21}$$

and

$$E^w \leq \vartheta^n E_0,$$

then

$$\sup_{B_{r\vartheta}(0)} d_{\mathbf{H}}(w, \hat{l}) \leq C r^{1+\alpha} \sup_{B_{R\vartheta}(0)} d_{\mathbf{H}}(w, L) + C r \vartheta \varphi_0^2, \ \forall r \in (0, \frac{R}{2}]$$

where $\hat{l} = (\hat{l}_\varrho, \hat{l}_\varphi) : B_1(0) \to \overline{\mathbf{H}}$ *is the almost affine map given by*

$$\hat{l}_\varrho(x) = w_\varrho(0) + \nabla w_\varrho(0) \cdot x, \quad \hat{l}_\varphi(x) = w_\varphi(x)$$

and $L : B_1(0) \to \overline{\mathbf{H}}$ *is any almost affine map.*

We also will need the following properties of the subset $\overline{\mathbf{H}}[\varphi_0, t_*]$.

Lemma 17 *Let* $\epsilon_0 \in (0,1)$*,* $D_0 \in (0, \frac{\epsilon_0}{2})$*,* $\theta \in (0,1]$ *and* $i = 0, 1, 2, \ldots$*. If* $Q = (\varrho, \varphi)$ *satisfies*

$$d_{\mathbf{H}}(Q, \overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*]) \leq \frac{\theta^i D_0}{2^i},$$

then

$$|\varrho| \leq \theta^i \epsilon_0 \quad \textit{or} \quad Q \in \overline{\mathbf{H}}[\ 2\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*].$$

Proof. Let $Q = (\varrho, \varphi)$ with

$$d_{\mathbf{H}}(Q, \overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*]) \leq \frac{\theta^i D_0}{2^i} \quad \text{and} \quad |\varrho| \geq \theta^i \epsilon_0.$$

With an intent of arriving at a contradiction, assume

$$Q \notin \overline{\mathbf{H}}[\ 2\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*]$$

and let $\gamma = (\gamma_\varrho, \gamma_\varphi) : [0,1] \to \overline{\mathbf{H}}$ be a geodesic with

$$\gamma(0) = Q \quad \text{and} \quad \gamma(1) \in \partial\overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*]$$

where $\gamma(1)$ is the point in $\overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*]$ closest to Q. We first claim

$$\min_{t \in [0,1]} |\gamma_\varrho(t)| \geq \frac{\theta^i \epsilon_0}{2}. \tag{22}$$

Indeed, assume on the contrary that $\gamma_\varrho(t_0) < \frac{\theta^i \epsilon_0}{2}$ for some $t_0 \in (0,1]$. Then since $\gamma_\varrho(0) \geq \theta^i \epsilon_0$, we obtain

$$\begin{aligned}
\frac{\theta^i \epsilon_0}{2} &< |\gamma_\varrho(t_0) - \gamma_\varrho(0)| \\
&\leq \int_0^{t_0} \left|\frac{d\gamma_\varrho}{dt}\right| dt \leq \int_0^{t_0} \left|\frac{d\gamma}{dt}\right| dt \\
&\leq d_{\mathbf{H}}(Q, \overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}], t_*) \leq \frac{\theta^i D_0}{2^i}.
\end{aligned}$$

This contradicts the assumption that $D_0 \in (0, \frac{\epsilon_0}{2})$ and proves (22). Combined with (18), we conclude

$$\left(\frac{\theta^i \epsilon_0}{2}\right)^3 \leq \mathcal{J}(\gamma(t)).$$

Therefore

$$\begin{aligned}
\left(\frac{\theta^i \epsilon_0}{2}\right)^3 \left| |\varphi| - \left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i} \right| &\leq \left(\frac{\theta^i \epsilon_0}{2}\right)^3 \int_0^1 \left|\frac{d\gamma_\varphi}{dt}(t)\right| dt \\
&\leq \int_0^1 \sqrt{\mathcal{J}(\gamma(t)) \left|\frac{d\gamma_\varphi}{dt}(t)\right|^2}\, dt \\
&\leq \int_0^1 \sqrt{\left|\frac{d\gamma_\varrho}{dt}(t)\right|^2 + \mathcal{J}(\gamma(t)) \left|\frac{d\gamma_\varphi}{dt}(t)\right|^2}\, dt \\
&= \text{length}(\gamma) \\
&= d_{\mathbf{H}}(Q, \gamma(1)) \\
&\leq \frac{\theta^i D_0}{2^i}
\end{aligned}$$

which in turn implies

$$|\varphi| \leq 2 \left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i},$$

In other words,

$$Q \in \overline{\mathbf{H}}[\ 2 \left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*].$$

This contradiction proves the assertion. Q.E.D.

To prove regularity of harmonic maps, [GS] introduces the notion of a homogeneous map being *effectively contained* in a totally geodesic subspace. The following lemma plays an analogous role in our setting.

Lemma 18 *Fix $\theta \in (0, \frac{1}{24})$. Given $A > 0$, $\epsilon_0 > 0$ $D_0 \in (0, \frac{\epsilon_0}{2})$ and $i \in \{0, 1, 2, \dots\}$, if*

$$ {}_i l : B_{\theta^i}(0) \to \overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*] $$

and

$$ v : B_{\theta^i}(0) \to \overline{\mathbf{H}} $$

satisfies

$$ \sup_{B_{\theta^i}(0)} |v_\varrho - Ax^1| < \theta^i \epsilon_0 \tag{23} $$

and

$$ \sup_{B_{\theta^i}(0)} d_{\mathbf{H}}(v, {}_i l) < \frac{\theta^i D_0}{2^i}, \tag{24} $$

then

$$ Vol\left\{ x \in B_{\theta^i}(0) : v(x) \notin \overline{\mathbf{H}}[2\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*] \right\} < \theta^{in} \frac{2\epsilon_0}{A}. $$

where Vol is the volume with respect to Euclidean metric.

Proof. Since ${}_i l(x) \in \overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*]$, assumption (24) implies that we have for $x \in B_{\theta^i}(0)$

$$ d_{\mathbf{H}}(v(x), \overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*]) \le \sup_{B_{\theta^i}(0)} d_{\mathbf{H}}(v, {}_i l) < \frac{\theta^i D_0}{2^i}. $$

Thus, Lemma 17 implies that

$$ \left\{ x \in B_{\theta^i}(0) : v(x) \notin \overline{\mathbf{H}}[2\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*] \right\} \subset \{ x \in B_{\theta^i}(0) : |v_\varrho(x)| \le \theta^i \epsilon_0 \}. $$

Furthermore, assumption (23) implies

$$|v_\varrho(x)| \le \theta^i \epsilon_0 \quad \Rightarrow \quad |Ax^1| \le |Ax^1 - v_\varrho(x)| + |v_\varrho(x)| < 2\theta^i \epsilon_0$$

in $B_{\theta^i}(0)$. Hence

$$\{x \in B_{\theta^i}(0) : |v_\varrho(x)| \le \theta^i \epsilon_0\} \subset \{x \in B_{\theta^i}(0) : |Ax^1| < 2\theta^i \epsilon_0\}.$$

The assertion now follows from the fact that

$$Vol\{x \in B_{\theta^i}(0) : |Ax^1| < 2\theta^i \epsilon_0\} \le \theta^{in} \frac{2\epsilon_0}{A}.$$

Q.E.D.

2.4 Harmonic maps into the Weil-Petersson completion $\overline{\mathcal{T}}$

Let $\mathcal{T} = \mathcal{T}(S)$ be the Teichmüller space of an oriented surface S of genus g and p marked points such that $k = 3g - 3 + p > 0$ and $\overline{\mathcal{T}}$ be the Weil-Petersson completion of $\mathcal{T}$. The complex dimension of $\mathcal{T}$ is $k = 3g - 3 - p$. The space $\overline{\mathcal{T}}$ is a stratified space; more precisely, we can write

$$\overline{\mathcal{T}} = \bigcup \mathcal{T}'$$

where $\mathcal{T}' = \mathcal{T}$ or $\mathcal{T}'$ is a lower dimensional Teichmüller space corresponding to the original surface with a number of curves pinched (see [Ab] or [Wo1] for further details). Recall that all the strata are totally geodesic with respect to the Weil-Petersson distance. For our purposes, a local model for the stratification is described as follows: Given a point $P \in \overline{\mathcal{T}}$, let P be in a stratum $\mathcal{T}'$ of complex dimension $j \in \{1, \ldots, k\}$. First notice that the stratification of $\overline{\mathbf{H}} = \mathbf{H} \cup \{P_0\}$ induces a stratification on the product space $\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}$. There exists a neighborhood $\mathcal{N} \subset \overline{\mathcal{T}}$ of P, a neighborhood $\mathcal{U} \subset \mathbf{C}^j$ of $\mathcal{O} = (0, \ldots, 0)$, a neighborhood $\mathcal{V} \subset \overline{\mathbf{H}}^{k-j}$ of $\mathcal{P}_0 = (P_0, \ldots, P_0)$ and a *stratification preserving* homeomorphism

$$\Psi : \mathcal{N} \to \mathcal{U} \times \mathcal{V} \subset \mathbf{C}^j \times \overline{\mathbf{H}}^{k-j} \tag{25}$$

such that

(i) Ψ induces a diffeomorphism on each stratum.

(ii) $\Psi(P) = (\mathcal{O}, \mathcal{P}_0) = (0, \ldots, 0, P_0, \ldots, P_0) \in \mathcal{U} \times \mathbf{H}^{k-j}$.

(iii) There exists a hermitian metric G along each stratum of $\mathcal{U} \times \mathcal{V}$ such that
$$\Psi : (\mathcal{N}, G_{WP}) \to (\mathcal{U} \times \mathcal{V}, G)$$
is a hermitian isometry between stratified spaces, where G_{WP} denotes the Weil-Petersson metric on each stratum of $\overline{\mathcal{T}}$.

For a map $u : \Omega \to \overline{\mathcal{T}}$, we define its *regular set* and *singular set* as

$$\mathcal{R}(u) = \{x \in \Omega : \exists r > 0 \text{ such that } B_r(u(x)) \subset \mathcal{T}' \text{ for a stratum } \mathcal{T}' \text{ of } \overline{\mathcal{T}}\}$$

and
$$\mathcal{S}(u) = \Omega \backslash \mathcal{R}(u).$$
A point in $\mathcal{R}(u)$ is called a *regular point* and a point in $\mathcal{S}(u)$ is called a *singular point.*

We define $\# : \overline{\mathcal{T}} \to \{1, \ldots, k\}$ by setting
$$\#P = j$$
and we say
$$(\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G) \text{ is a } \textit{local model at } P \in \overline{\mathcal{T}}. \tag{26}$$
We can decompose the singular set $\mathcal{S}(u)$ of a harmonic map $u : \Omega \to \overline{\mathcal{T}}$ as a disjoint union of sets
$$\mathcal{S}(u) = \bigcup_{j=0}^{k} \hat{\mathcal{S}}_j(u)$$
where
$$\hat{\mathcal{S}}_j(u) = \{x \in \mathcal{S}(u) : \#u(x) = j\}, \quad j = 1, \ldots, k.$$
Furthermore, let
$$\mathcal{S}_j(u) = \hat{\mathcal{S}}_j(u) \backslash \mathcal{S}_0(u) \tag{27}$$
where recall that $\mathcal{S}_0(u)$ is the set of higher order points. In other words, $\mathcal{S}_j(u)$ is the set of order 1 singular points of $\hat{\mathcal{S}}_j(u)$. If $\#P = k$, then $P \in \mathcal{T}$, and hence $P \in \mathcal{R}(u)$. Thus, $\hat{\mathcal{S}}_k(u) = \mathcal{S}_k(u) = \emptyset$.

If $x_\star \in \hat{\mathcal{S}}_j(u)$, then there exists $\sigma_\star > 0$ such that we can represent the restriction map $u|_{B_{\sigma_\star}(x_\star)}$ as

$$u = (V, v) : (B_{\sigma_\star}(x_\star), g) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G). \tag{28}$$

We observe that by definition,

$$\hat{\mathcal{S}}_l(u) \cap B_{\sigma_\star}(x_\star) = \mathcal{S}_l(u) \cap B_{\sigma_\star}(x_\star) = \emptyset, \quad \forall l = 1, \dots, k-j+1. \tag{29}$$

Using the natural identification $\mathcal{U} = \mathcal{U} \times \{P_0\}$ we extend to $\mathbf{C}^j$ the pullback

$$H := \Psi^{-1}|^*_{\mathcal{U}_{P_0}} G_{WP}$$

defined on $\mathcal{U} \subset \mathbf{C}^j$, where in the above G_{WP} denotes the Weil-Petersson metric on the stratum $\mathcal{T}'$. Furthermore, we denote by h the product metric on any product $\mathbf{H}^l$; i.e.

$$(\mathbf{H}^l, h) = (\mathbf{H} \times \ldots \times \mathbf{H}, g_{\mathbf{H}} \oplus \ldots \oplus g_{\mathbf{H}})$$

which induces a metric on the stratified space $\overline{\mathbf{H}}^{k-j}$. The *regular component map* of u is the map

$$V : (B_{\sigma_\star}(x_\star), g) \to (\mathbf{C}^j, H)$$

into the hermitian manifold $(\mathbf{C}^j, H)$. The *singular component* of u is the map

$$v = (v^1, \dots, v^{k-j}) : (B_{\sigma_\star}(x_\star), g) \to (\overline{\mathbf{H}}^{k-j}, d)$$

into the NPC space $(\overline{\mathbf{H}}^{k-j}, d)$ where d be distance function on $\overline{\mathbf{H}}^{k-j}$ induced by the hermitian metric $h = g_{\mathbf{H}} \oplus \ldots \oplus g_{\mathbf{H}}$ on $\mathbf{H}^{k-j}$.

2.5 Harmonic maps into asymptotically product spaces

In [DM1], we developed a general technique for studying harmonic maps into special NPC spaces that are asymptotically a product space. There, we are specifically interested in "differentiable manifold connected complexes" or DM-complexes for short (generalizing the "flat connected" or *F-connected* complexes of [GS]). A DM-complex Y is characterized by the property that every two points of Y is contained in a DM, or a differentiable manifold with

a Riemannian metric, isometrically embedded in Y. In [DM1], we proved the corresponding statements of Theorem 1 and Theorem 2 for a harmonic map into a DM-complex with the singular set defined as the complement of the set of points whose neighborhood is mapped to a single DM.

On the other hand, as discussed in [DM1], the theory works in a more general context, and we will apply it to prove Theorem 1 and Theorem 2. Below, we list the assumptions needed in [DM1] for a local representation

$$u = (V, v) : (B_{\sigma_\star}(x_\star), g) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G)$$

of a harmonic map into $\overline{\mathbf{H}}$. In Section 5, we apply the inductive argument (with respect to the integer j) of [DM1] by showing that at each step of the induction, these assumptions hold.

Assumption 1 The NPC space $\overline{\mathbf{H}}^{k-j}$ (endowed with product distance function d) has a *homogeneous structure* with respect to a base point $\mathcal{P}_0 = (P_0, \dots, P_0) \in \overline{\mathbf{H}}^{k-j}$. In other words, there is a continuous map

$$\mathbf{R}_{>0} \times \overline{\mathbf{H}}^{k-j} \to \overline{\mathbf{H}}^{k-j}, \quad (\lambda, P) \mapsto \lambda P$$

such that $\lambda \mathcal{P}_0 = \mathcal{P}_0$ for every $\lambda > 0$ and the distance function d is homogeneous of degree 1 with respect to this map, i.e.

$$d(\lambda P, \lambda P') = \lambda d(P, P'), \ \forall P, P' \in \overline{\mathbf{H}}^{k-j}.$$

Assumption 2 The metric H on $\mathbf{C}^j$ and the metric h of $\overline{\mathbf{H}}^{k-j}$ are such that the metric G is asymptotically the product metric $H \oplus h$ in the following sense:

First note that, any holomorphic coordinate system v on $\mathbf{H}$ induces coordinates on the stratified space $\overline{\mathbf{H}}^{k-j}$. With this understood, there exist constants $C > 0$, $\epsilon \in (0, \frac{1}{2})$ and holomorphic coordinates v on $\mathbf{H}$ such that if, with respect to the standard coordinates $(V^1, \dots, V^j)$ of $\mathbf{C}^j$ and the coordinates $(v^{j+1}, \dots, v^k)$ of $\mathbf{H}^{k-j}$ induced by v, we write

$$H(V) = (H_{I\overline{L}}(V)), \qquad H^{-1}(V) = (H^{I\overline{L}}(V)),$$
$$h(v) = (h_{i\bar{l}}(v)), \qquad h^{-1}(v) = (h^{i\bar{l}}(v)),$$
$$G(V,v) = \begin{pmatrix} G_{I\overline{L}}(V,v) & G_{I\bar{l}}(V,v) \\ G_{i\overline{L}}(V,v) & G_{i\bar{l}}(V,v) \end{pmatrix}, \qquad G^{-1}(V,v) = \begin{pmatrix} G^{I\overline{L}}(V,v) & G^{I\bar{l}}(V,v) \\ G^{i\overline{L}}(V,v) & G^{i\bar{l}}(V,v) \end{pmatrix}$$

with $I, L = 1, \ldots, j$ and $i, l = j+1, \ldots, k$ then the following estimates hold:

C^0-estimates:

$$\begin{array}{rcl} |G_{I\overline{L}}(V,v) - H(V)_{I\overline{L}}| & \leq & CH(V)^{\frac{1}{2}}_{I\overline{I}} H(V)^{\frac{1}{2}}_{L\overline{L}} d^2(v, P_0) \\ |G_{I\bar{l}}(V,v)| & \leq & CH(V)^{\frac{1}{2}}_{I\overline{I}} h(v)^{\frac{1}{2}}_{l\bar{l}}\ d^2(v, P_0) \\ |G_{i\bar{l}}(V,v) - h_{i\bar{l}}(v)| & \leq & Ch(v)^{\frac{1}{2}}_{i\bar{i}} h(v)^{\frac{1}{2}}_{l\bar{l}}\ d^2(v, P_0) \end{array} \tag{30}$$

C^1-estimates:

$$\begin{array}{rcl} |\frac{\partial}{\partial V^I} G_{L\overline{K}}(V,v)| & \leq & CH(V)^{\frac{1}{2}}_{I\overline{I}} H(V)^{\frac{1}{2}}_{L\overline{L}} H(V)^{\frac{1}{2}}_{K\overline{K}} \\ |\frac{\partial}{\partial v^l} G_{I\overline{L}}(V,v)| & \leq & Ch(v)^{\frac{1}{2}}_{l\bar{l}} H(V)^{\frac{1}{2}}_{I\overline{I}} H(V)^{\frac{1}{2}}_{L\overline{L}} d(v, P_0) \\ |\frac{\partial}{\partial V^I} G_{L\bar{l}}(V,v)| & \leq & CH(V)^{\frac{1}{2}}_{I\overline{I}} H(V)^{\frac{1}{2}}_{L\overline{L}} h(v)^{\frac{1}{2}}_{l\bar{l}}\ d(v, P_0) \\ |\frac{\partial}{\partial v^m} G_{I\bar{l}}(V,v)| & \leq & CH(V)^{\frac{1}{2}}_{I\overline{I}} h(v)^{\frac{1}{2}}_{m\overline{m}} h(v)^{\frac{1}{2}}_{l\bar{l}} \\ |\frac{\partial}{\partial V^L} G_{i\bar{l}}(V,v)| & \leq & CH(V)^{\frac{1}{2}}_{L\overline{L}} h(v)^{\frac{1}{2}}_{i\bar{i}} h(v)^{\frac{1}{2}}_{l\bar{l}} \\ |\frac{\partial}{\partial v^m} \left(G_{i\bar{l}}(V,v) - h_{i\bar{l}}(v)\right)| & \leq & Ch(v)^{\frac{1}{2}}_{m\overline{m}} h(v)^{\frac{1}{2}}_{i\bar{i}} h(v)^{\frac{1}{2}}_{l\bar{l}} \end{array} \tag{31}$$

C^0-estimates of the inverse:

$$\begin{array}{rcl} |G^{I\overline{L}}(V,v) - H^{I\overline{L}}(V)| & \leq & CH^{I\overline{I}}(V)^{\frac{1}{2}} H^{L\overline{L}}(V)^{\frac{1}{2}} d^2(v, P_0) \\ |G^{I\bar{l}}(V,v)| & \leq & CH^{I\overline{I}}(V)^{\frac{1}{2}} h^{l\bar{l}}(v)^{\frac{1}{2}}\ d^2(v, P_0) \\ |G^{i\bar{l}}(V,v) - h^{i\bar{l}}(v)| & \leq & Ch^{i\bar{i}}(v)^{\frac{1}{2}} h^{l\bar{l}}(v)^{\frac{1}{2}}\ d^2(v, P_0) \end{array} \tag{32}$$

Almost diagonal condition for H and h with respect to the coordinates $(V^1, \ldots, V^j)$ and $(v^{j+1}, \ldots, v^k)$*:*

$$\begin{array}{cc} H_{I\overline{L}}(V) \leq \epsilon H_{I\overline{I}}(V)^{\frac{1}{2}} H_{L\overline{L}}(V)^{\frac{1}{2}} (I \neq L), & h_{i\bar{l}}(v) \leq \epsilon h_{i\bar{i}}(v)^{\frac{1}{2}} h_{l\bar{l}}(v)^{\frac{1}{2}} (i \neq l) \\ H_{I\overline{I}}(V) H^{I\overline{I}}(V) \leq C, & h_{i\bar{i}}(v) h^{i\bar{i}}(v) \leq C \end{array} \tag{33}$$

Bounds on the derivatives for H and h:

$$\begin{array}{rcl} |\frac{\partial}{\partial V^I} H_{L\overline{K}}(V)| & \leq & CH_{I\overline{I}}(V)^{\frac{1}{2}} H_{L\overline{L}}(V)^{\frac{1}{2}} H_{K\overline{K}}(V)^{\frac{1}{2}} \\ d(v, P_0)|\frac{\partial}{\partial v^i} h_{l\overline{m}}| & \leq & Ch_{i\bar{i}}(v)^{\frac{1}{2}} h_{l\bar{l}}(v)^{\frac{1}{2}} h_{m\overline{m}}(v)^{\frac{1}{2}}. \end{array} \tag{34}$$

Assumption 3 The set $\mathcal{S}_j(u)$ satisfies the following:

(*i*) $v(x) = \mathcal{P}_0$ *for* $x \in \mathcal{S}_j(u) \cap B_{\sigma_\star}(x_\star)$

(ii) $\dim_{\mathcal{H}}((\mathcal{S}(u)\backslash\mathcal{S}_j(u)) \cap B_{\frac{\sigma_\star}{2}}(x_\star)) \leq n-2.$

Assumption 4 For $B_R(x_0) \subset B_{\frac{\sigma_\star}{2}}(x_\star)$ and any harmonic map

$$w : (B_R(x_0), g) \to (\overline{\mathbf{H}}^{k-j}, d),$$

denote by $\mathcal{R}(u,w)$ the set of points x with the property that there exists $r > 0$ such that there exists a stratum Σ of $\overline{\mathbf{H}}^{k-j}$ such that for every $P_0 \in v(B_r(x))$ and $P_1 \in w(B_r(x))$ $P_t = (1-t)P_0 + tP_1 \in \Sigma$ for all $t \in (0,1)$. As usual, the sum here indicates geodesic interpolation. Then $\mathcal{R}(u,w)$ is of full measure in $\mathcal{R}(u) \cap B_R(x_0)$.

By [DM1] Remark 32, the results of [DM1] Section 6 are valid whenever Assumptions 1-4 are satisfied. The main result of that section is

Theorem 19 (The First Variation Formula for the Target) *Let*

$$u = (V, v) : (B_{\sigma_\star}(x_\star), g) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G)$$

be a harmonic map as in (28). Given $R \in (0, \sigma_*)$, a harmonic map

$$w : B_R(0) \to \overline{\mathbf{H}}^{k-j}$$

and a non-negative smooth function η with compact support in $B_R(x_\star)$, define

$$v_{t\eta}(x) = (1 - t\eta(x))v(x) + t\eta(x)w(x)$$

where the sum indicates geometric interpolation in $\overline{\mathbf{H}}^{k-j}$. If Assumptions 1-4 are satisfied, then there exists $\sigma_0 > 0$ and $C > 0$ such that

$$\limsup_{t\to 0^+} \frac{E^v(\sigma) - \ E^{v_{t\eta}}(\sigma)}{t} \leq C \int_{B_\sigma(0)} \eta(d(v, P_0) + |\nabla v|) d(v,w) d\mu \qquad (35)$$

for $\sigma \in (0, \sigma_0]$. Furthermore, C and σ_0 depend only on the constant in the estimates (30), (31), (32), (33) and (34) of the metric G, the domain metric g, the Lipschitz constant of u in $B_R(0)$.

PROOF. The proof is contained in [DM1] with only minor differences due to the fact that in our case the strata are not necessarily closed. Indeed, the closeness assumption of the strata in [DM1] can be easily replaced by the fact that the interior of a geodesic in $\overline{\mathbf{H}}^{k-j}$ connecting two points depends smoothly on the end points. With the notation as in [DM1] Section 6, this implies the following facts needed in the proof: If Σ is a stratum of $\overline{\mathbf{H}}^{k-j}$ and Σ_0 and Σ_1 are two strata of $\overline{\Sigma}$ (i.e. $\Sigma_0, \Sigma_1 \subset \partial\Sigma$) such that $v(B_r(x)) \subset \Sigma_0$ and $w(\mathcal{R}(w) \cap B_r(x)) \subset \Sigma_1$, then the geodesic interpolation

$$v : B_r(x) \times [0,1] \to \overline{\mathbf{H}}^{k-j}, \quad v(y,t) = (1-t)v(y) + tw(y)$$

satisfies the following properties:

(1) The point $v_{t\eta}(y) = v(y, t\eta)$ is in Σ for all $x \in B_r(x)$ and $t \in (0,1)$ where $\eta \in C^\infty(B_r(x))$, $0 \le \eta(x) \le 1$.

(2) For any $y \in B_r(x)$, the function $t \mapsto h_{ii}^{\frac{1}{2}}(v_{t\eta})\frac{\partial v^i_{t\eta}}{\partial x^\beta}(y)$ is continous in $t \in [0,1)$,

(3) For any $y \in B_r(x)$, the function $t \mapsto h_{ii}^{\frac{1}{2}}(v_{t\eta})\frac{\partial v^i_{t\eta}}{\partial t}(y)$ is continous in $t \in [0,1)$ and

(4) For any $y \in B_r(x)$, the function $t \mapsto d(v,\mathcal{P}_0) h_{ii}^{\frac{1}{2}}(v_{t\eta})\frac{d}{dt}\frac{\partial v^i_{t\eta}}{\partial x^\beta}(y)$ converges as $t \to 0^+$ to

$$d(v,\mathcal{P}_0) h_{ii}^{\frac{1}{2}}(v_{t\eta}) \liminf_{\tau \to 0^+} \frac{\frac{\partial v^i_{t\eta}}{\partial x^\beta}(y,\tau) - \frac{\partial v^i_0}{\partial x^\beta}(y,0)}{\tau}.$$

Given these facts, the proof of the proposition now follows as in [DM1]. Q.E.D.

Additionally, by [DM1] Remark 42, the results of [DM1] Section 7 are valid whenever Assumption 1-4 are satisfied. In particular, we have the following.

Lemma 20 *Let*

$$u = (V, v) : (B_{\sigma_\star}(x_\star), g) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G)$$

be a harmonic map as in (28). For $x_0 = 0 \in B_{\frac{\sigma_\star}{2}}(x_\star) \cap \mathcal{S}_j(u)$, $B_{\sigma_0}(0) \subset B_{\frac{\sigma_\star}{2}}(x_\star)$ and v_σ be the blow up map of v at $x_0 = 0$ where the blow up factor $\nu(\sigma)$ is equal to either $\sqrt{\frac{I^v(\sigma)}{\sigma^{n-1}}}$ or $\sqrt{\frac{I^u(\sigma)}{\sigma^{n-1}}}$. Assume $E^{v_\sigma}(1) \leq A$. For $r \in (0,1)$, there exists a constant $C > 0$ depending only on r, the constant in the estimates (30)-(34) for the target metric G, the domain metric g, the Lipschitz constant of u and A such that for any $\sigma \in (0, \sigma_0)$ and any harmonic map

$$w : (B_\vartheta(0), g_\sigma) \to \overline{\mathbf{H}}^{k-j}$$

with $E^w(1) \leq E^{v_\sigma}(1)$, we have

$$\sup_{B_{\vartheta r}(0)} d^2(v_\sigma, w) \leq C \int_{\partial B_\vartheta(0)} d^2(v_\sigma, w) d\Sigma_\sigma + C\sigma\vartheta^3. \tag{36}$$

Proof. In the first case when $\nu(\sigma)$ is equal to $\sqrt{\frac{I^v(\sigma)}{\sigma^{n-1}}}$, inequality (36) follows immediately from [DM1], Lemma 48. The second case when $\nu(\sigma) = \sqrt{\frac{I^u(\sigma)}{\sigma^{n-1}}}$ is similar. Q.E.D.

Assumption 5 For almost every $x \in \mathcal{S}_j(u)$, we have

$$|\nabla v|^2(x) = 0 \quad \text{and} \quad |\nabla V|^2(x) = |\nabla u|^2(x).$$

Assumption 6 For any subdomain Ω compactly contained in

$$B_{\frac{\sigma_\star}{2}}(x_\star) \backslash \left(\mathcal{S}(u) \cap v^{-1}(\mathcal{P}_0) \right),$$

there exists a sequence of smooth functions $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $\mathcal{S}(u) \cap \overline{\Omega}$, $0 \leq \psi_i \leq 1$, $\psi_i \to 1$ for all $x \in \Omega \backslash \mathcal{S}(u)$ such that

$$\lim_{i \to \infty} \int_{B_{\frac{\sigma_\star}{2}}(x_\star)} |\nabla\nabla u||\nabla \psi_i| \, d\mu = 0.$$

Following [DM1], we can show that under the above assumptions, the singular component map v of a local representation has a well-defined order and a tangent map. More precisely, we have the following statement which should be compared to Remark 7 and Remark 8.

Theorem 21 *If the metric G on $\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}$, H on $\mathbf{C}^j$, h on $\overline{\mathbf{H}}^{k-j}$ and the local representation*

$$u = (V, v) : (B_{\sigma_\star}(x_\star), g) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G)$$

satisfies Assumptions 1-6, then v satisfies $(P1)$ and $(P2)$ with respect $\mathcal{S}_j(u)$ as in Definition 9. Furthermore, there exists $c > 0$ such that

$$\sigma \mapsto e^{c\sigma} \frac{I_0^v(\sigma)}{\sigma^{n-1+2\alpha}} \quad \textit{and} \quad \sigma \mapsto e^{c\sigma} \frac{E_0^v(\sigma)}{\sigma^{n-2+2\alpha}} \tag{37}$$

are monotone non-decreasing functions where $\alpha = Ord^v(x_0)$.

3 Asymptotically harmonic maps

The singular component map $v : B_R(0) \to \overline{\mathbf{H}}^{k-j}$ of a local representation (28) is not necessarily harmonic. However, we will later prove that blow up maps of the singular component map v of a harmonic map u into $\overline{\mathcal{T}}$ satisfy the following definition.

Definition 22 We say that a sequence of maps $\{v_i : (B_1(0), g_i) \to \overline{\mathbf{H}}^{k-j}\}$ is a sequence of *asymptotically harmonic maps* if the following conditions are satisfied:

- (i) The sequence of metrics $\{g_i\}$ on $B_1(0) \subset \mathbf{R}^n$ converges in C^k for any $k = 1, 2, \ldots$ to the Euclidean metric g_0 on $B_1(0) \subset \mathbf{R}^n$.
- (ii) For $R \in (0, 1)$, there exists a constant $E_0 > 0$ such that $E^{v_i}(\vartheta) \leq \vartheta^n E_0$ for $\vartheta \in (0, R]$.
- (iii) The sequence $\{v_i\}$ converges locally uniformly in the pullback sense to a homogeneous harmonic map $v_0 : (B_1(0), g_0) \to (Y_0, d_0)$ into an NPC space. (Note that because $\overline{\mathbf{H}}^{k-j}$ is not locally compact, we cannot assume that Y_0 is $\overline{\mathbf{H}}^{k-j}$).
- (iv) For a fixed $R \in (0, 1)$ and $r \in (0, 1)$, there exist $c_0 > 0$ and a sequence $c_i \to 0$ such that for any harmonic map $w : (B_R(0), g_i) \to \overline{\mathbf{H}}^{k-j}$ with $E^w(R) \leq E^{v_i}(R)$, we have

$$\sup_{B_{r\vartheta}(0)} d^2(v_i, w) \leq \frac{c_0}{\vartheta^{n-1}} \int_{\partial B_\vartheta(0)} d^2(v_i, w) d\Sigma + c_i \vartheta^3, \ \forall \vartheta \in (0, R]. \tag{38}$$

Remark 23 The subsequence of blow up maps in Remark 8 is a sequence of asymptotically harmonic maps. In particular, since u_{σ_i} is harmonic for each i, inequality (38) is satisfied with $v_i = u_{\sigma_i}$ and $c_i = 0$ (cf. [KS1] Lemma 2.4.2).

The next lemma is one of the key technical ingredients in this paper. We will it later apply it to a blow up map of a singular component map of a harmonic map into $\overline{\mathcal{T}}$. As noted in the introduction, its predecessor is [GS] Theorem 5.1.

Inductive Lemma 24 *Given $c_0 \geq 1$, $E_0, A^1, \dots, A^m > 0$, there exist $\theta \in (0, \frac{1}{24})$, $\epsilon_0 > 0$ and $D_0 > 0$ that satisfy the following statement.*

Assume the following:

- *The map*

$$l = (l^1 \circ (R^1)^{-1}, \dots, l^m \circ (R^m)^{-1}, l^{m+1}, \dots, l^{k-j}) : B_{\theta^i}(0) \to \overline{\mathbf{H}}^{k-j}$$

  is such that R^μ is a rotation,

  $l^\mu(x) = (A^\mu x^1, 0)$ in coordinates (ϱ, φ) anchored at $t_*^\mu \in (-\infty, \frac{3}{2})$ (cf. (15))

  for $\mu = 1, \dots, m$ and

$$l^\mu \text{ is identically equal to } P_0$$

  for $\mu = m+1, \dots, k-j$.

- *The subset $\overline{\mathbf{H}}[2\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*^\mu]$ satisfies*

$$a[2\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*^\mu] = a[\frac{16 D_0}{\epsilon_0^3 \theta^{2i} 2^i}, t_*^\mu] < \frac{\theta^i}{2} \quad \text{(cf. (20))} \tag{39}$$

  for $\mu = 1, \dots, m$.

- *The map*

$$v = (v^1, \dots, v^{k-j}) : (B_1(0), g) \to \overline{\mathbf{H}}^{k-j}$$

  is such that g is a normalized metric,

$$v(0) = \mathcal{P}_0, \quad E^v(\vartheta) \leq \vartheta^n E_0 \tag{40}$$

  and

for $R \in (0, \frac{7}{8}]$, a harmonic map $w : (B_{\theta^i R}(0), g) \to \overline{\mathbf{H}}^{k-j}$ with $E^w(\theta^i R) \leq E^v(\theta^i R)$ and a constant

$$c = \frac{\theta^2 D_0^2}{2^8}, \tag{41}$$

we have

$$\sup_{B_{\frac{15\theta^i R}{16}}(0)} d^2(v, w) \leq \frac{c_0}{(\theta^i R)^{n-1}} \int_{\partial B_{\theta^i R}(0)} d^2(v, w) d\Sigma + c\theta^{3i}. \tag{42}$$

- *The metric g is sufficiently close to the Euclidean metric such that if we denote Vol and Vol_g to be the volume with respect to the standard Euclidean metric and the metric g respectively, then*

$$\frac{15}{16} Vol(S) \leq Vol_g(S) \leq \frac{17}{16} Vol(S) \tag{43}$$

for any smooth submanifold S of $B_1(0)$.

- *The map*

$${}_i l = ({}_i l^1 \circ (R^1)^{-1}, \dots, {}_i l^m \circ (R^1)^{-1}, {}_i l^{m+1}, \dots, {}_i l^{k-j}) : B_{\theta^i}(0) \to \overline{\mathbf{H}}^{k-j},$$

is such that

$${}_i l^\mu : B_{\theta^i}(0) \to \overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*^\mu] \text{ is an almost affine map}$$

for $\mu = 1, \dots, m$ (cf. Definition 15) and

$${}_i l^\mu \text{ is identically equal to } P_0$$

for $\mu = m+1, \dots, k-j$.

- *The constant ${}_i\delta > 0$ is such that*

$$\begin{cases} \sup_{B_{\theta^i}(0)} d(v, {}_i l) < \theta^i \dfrac{D_0}{2^i} \\ \sup_{B_{\theta^i}(0)} |v_\varrho^\mu \circ R^\mu(x) - A^\mu x^1| < \theta^i {}_i\delta < \theta^i \displaystyle\sum_{k=0}^{i} \frac{\theta^{-1} D_0}{2^{k-2}}. \end{cases} \tag{44}$$

Then there exists a map

$${}_{i+1}l = ({}_{i+1}l^1 \circ (R^1)^{-1}, \dots, {}_{i+1}l^m \circ (R^m)^{-1}, {}_{i+1}l^{m+1}, \dots, {}_{i+1}l^{k-j}) : B_{\theta^{i+1}}(0) \to \overline{\mathbf{H}}^{k-j}$$

such that

$${}_{i+1}l^\mu : B_{\theta^{i+1}}(0) \to \overline{\mathbf{H}}[\left(\frac{\theta^{i+1}\epsilon_0}{2}\right)^{-3} \frac{\theta^{i+1} D_0}{2^{i+1}}, t^\mu_*] \text{ is an almost affine map}$$

for $\mu = 1, \dots, m$,

$${}_{i+1}l^\mu \text{ is identically equal to } P_0$$

for $\mu = m+1, \dots, k-j$ *and*

$$\begin{cases} \sup\limits_{B_{\theta^{i+1}}(0)} d(v, {}_{i+1}l) < \theta^{i+1} \dfrac{D_0}{2^{i+1}} \\ \sup\limits_{B_{\theta^{i+1}}(0)} |v^\mu_\rho \circ R^\mu(x) - A^\mu x^1| < {}_{i+1}\delta\theta^{i+1} := \left({}_i\delta + \dfrac{2D_0\theta^{-1}}{2^i}\right)\theta^{i+1} < \theta^{i+1} \sum\limits_{k=0}^{i+1} \dfrac{\theta^{-1} D_0}{2^{k-2}} \\ \sup\limits_{B_{\theta^{i+1}}(0)} d(v, l) < m\theta^{i+1} \left(\dfrac{2^3 (A + 9D_0)^3}{\epsilon_0^3} + 10\right) \theta^{-1} D_0. \end{cases} \tag{45}$$

Proof. Let

$$A_{\min} := \min\{A^1, \dots, A^m\} \text{ and } A_{\max} := \max\{A^1, \dots, A^m\}. \tag{46}$$

For $R = \frac{1}{2}$, $E_0 > 0$ as in (40), $A_0 = 4A_{\max}$ and the metric g given in the statement of the theorem, let

$$C \geq 1 \text{ and } \alpha > 0 \text{ be as in Theorem 16.} \tag{47}$$

Let $\theta \in (0, \min\{\frac{1}{24}, \frac{1}{\sqrt{A}}\})$ sufficiently small such that

$$C\theta < 1, \tag{48}$$

$$C\theta^\alpha < \frac{1}{2^6}, \tag{49}$$

and

$$C\theta^3 < \frac{1}{2^3}. \tag{50}$$

Define

$$\epsilon_0 := \left(\frac{A_{\min}}{2^{2n+11}c_0}\right)\theta^2 < 1. \tag{51}$$

Choose $D_0 > 0$ such that

$$D_0 < \min\left\{\frac{\epsilon_0^6}{2^{13}mC}, \frac{A_{\min}}{4}, \frac{\theta\epsilon_0}{8}\right\}. \tag{52}$$

Furthermore, inequality (52) implies $8\theta^{-1}D_0 < \epsilon_0$. Combining this with (44), we obtain

$$\sup_{B_{\theta^i}(0)} |v_\varrho^\mu \circ R^\mu(x) - A^\mu x^1| < \theta^i 8\theta^{-1}D_0 < \theta^i\epsilon_0.$$

Thus, the assumption (23) of Lemma 18 is satisfied. Additionally, the assumption (24) of Lemma 18 is implied by (44). Thus, Lemma 18 and (43) imply

$$Vol_g\left\{x \in B_{\theta^i}(0) : v^\mu(x) \circ R^\mu \notin \overline{\mathbf{H}}[2\left(\frac{\theta^i\epsilon_0}{2}\right)^{-3}\frac{\theta^i D_0}{2^i}, t_*^\mu]\right\} < \frac{17}{16}\cdot\theta^{in}\frac{2\epsilon_0}{A^\mu}$$

which in turn implies that there exists $R \in [\frac{5}{8}, \frac{7}{8}]$ with the property that

$$Vol_g\left\{x \in \partial B_{\theta^i R}(0) : v^\mu \circ R^\mu(x) \notin \overline{\mathbf{H}}[2\left(\frac{\theta^i\epsilon_0}{2}\right)^{-3}\frac{\theta^i D_0}{2^i}, t_*^\mu]\right\} < (\theta^i R)^{n-1}\frac{2^{2n+2}\epsilon_0}{A^\mu}. \tag{53}$$

Let

$$w = (w^1, \ldots, w^{k-j}) : B_{\theta^i R}(0) \to \overline{\mathbf{H}}^{k-j}$$

be the harmonic map defined as follows:

- Let

  $$\pi^\mu : \overline{\mathbf{H}} \to \overline{\mathbf{H}}[2\left(\frac{\theta^i\epsilon_0}{2}\right)^{-3}\frac{\theta^i D_0}{2^i}, t_*^\mu], \quad \mu = 1, \ldots, m$$

  be the closest point projection map and

  $$w^\mu : B_{\theta^i R}(0) \to \overline{\mathbf{H}}[2\left(\frac{\theta^i\epsilon_0}{2}\right)^{-3}\frac{\theta^i D_0}{2^i}, t_*^\mu], \quad \mu = 1, \ldots, m$$

  be the harmonic map with boundary value equal to $\pi^\mu \circ v^\mu$.

- For $\mu = m+1, \ldots, k-j$, let w^μ be identically equal to P_0.

By the definition of π^μ, the fact that ${}_il^\mu(x) \in \overline{\mathbf{H}}[\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t_*^\mu]$ for $\mu = 1, \ldots, m$ and that ${}_il^\mu(x) \equiv P_0$ for $\mu = m+1, \ldots, k-j$, we conclude

$$d(v(x), w(x)) \leq d(v(x), {}_il(x)), \quad \forall x \in B_{\theta^i R}(0). \tag{54}$$

Since $\theta < \frac{1}{24}$, we have $\theta^i < \frac{1}{2^{2i+1}}$, and thus (41) implies

$$c\theta^{3i} = c\theta^{-2}\theta^{3i+2} < \theta^{2i+2}\frac{D_0^2}{2^{2i+9}}. \tag{55}$$

We thus obtain

$$\begin{aligned}
\sup_{B_{\frac{15\theta^i R}{16}}(0)} d^2(v,w) &\leq \frac{c_0}{(\theta^i R)^{n-1}} \int_{\partial B_{\theta^i R}(0)} d^2(v,w) d\Sigma + c\theta^{3i} \quad \text{(by (42))} \\
&< \frac{2^{2n+2}\epsilon_0 c_0}{A_{\min}} \sup_{\partial B_{\theta^i R}(0)} d^2(v,w) + c\theta^{3i} \quad \text{(by (53))} \\
&\leq \frac{2^{2n+2}\epsilon_0 c_0}{A_{\min}} \sup_{\partial B_{\theta^i R}(0)} d^2(v, {}_il) + c\theta^{3i} \quad \text{(by (54))} \\
&< \frac{2^{2n+2}\epsilon_0 c_0}{A_{\min}} \cdot \theta^{2i}\frac{D_0^2}{2^{2i}} + c\theta^{3i} \qquad \text{(by (44))} \\
&< \theta^{2i+2}\frac{D_0^2}{2^{2i+8}} \qquad \text{(by (51) and (55))},
\end{aligned} \tag{56}$$

or more simply

$$\sup_{B_{\frac{15\theta^i R}{16}}(0)} d(v,w) < \theta^{i+1}\frac{D_0}{2^{i+4}}. \tag{57}$$

Combining (44) and (57), we obtain

$$\sup_{B_{\frac{\theta^i}{2}}(0)} d(w, {}_il) \leq \sup_{B_{\frac{\theta^i}{2}}(0)} d(v,w) + \sup_{B_{\frac{\theta^i}{2}}(0)} d(v, {}_il) \leq \theta^i \frac{D_0}{2^{i-1}}. \tag{58}$$

We will now check that we can apply Theorem 16. We fix $R = \frac{1}{2}$, E_0 as in (40), $A_0 = 4A_{\max}$. With $\varphi_0 = 2\left(\frac{\epsilon_0}{2}\right)^{-3}\frac{D_0}{2^i}$ and $\vartheta = \theta^i$, First, note that since a projection into a convex set in an NPC space is distance non-increasing,

we obtain $E^{w^\mu}(\theta^i) \leq E^{v^\mu}(\theta^i) \leq E^v(\theta^i) \leq \theta^{in} E_0$ by (40). Next, Lemma 14, (57), (44) and (52) imply that in $B_{\frac{15\theta^i R}{16}}(0)$, we have

$$\begin{aligned}
|w^\mu_\varrho \circ R^\mu| &\leq |w^\mu_\varrho \circ R^\mu - v^\mu_\varrho \circ R^\mu| + |v^\mu_\varrho \circ R^\mu - A^\mu x^1| + |A^\mu x^1| \\
&< \theta^{i+1} \frac{D_0}{2^{i+4}} + 2\theta^i D_0 + \theta^i A^\mu \\
&\leq 4\theta^i A^\mu \leq A_0 \vartheta.
\end{aligned}$$

Thus, w^μ maps into $\overline{\mathbf{H}}[\frac{\varphi_0}{\vartheta^2}, t_*] \cap \mathbf{B}_{A_0\vartheta}(P_0)$. Finally, (39) implies

$$a[\frac{\varphi_0}{\vartheta^2}, t^\mu_*] = a[2\left(\frac{\theta^i \epsilon_0}{2}\right)^{-3} \frac{\theta^i D_0}{2^i}, t^\mu_*] = a[\frac{16 D_0}{\epsilon_0^3 \theta^{2i} 2^i}, t^\mu_*] < \frac{\theta^i}{2} = \frac{\vartheta}{2}$$

which is assumption (21) of Theorem 16. Thus, with

$$ {}_i l^\mu = L, \ {}_{i+1}l^\mu = \hat{l}, \ \vartheta = \theta^i, \ R = \frac{1}{2} \text{ and } \ r = \theta$$

in Theorem 16, we have by the choice of the constants in (47) that

$$\sup_{B_{\theta^{i+1}}(0)} d_{\mathbf{H}}(w^\mu, {}_{i+1}l^\mu) \leq C\theta^{1+\alpha} \sup_{B_{\frac{\theta^i}{2}}(0)} d_{\mathbf{H}}(w^\mu, {}_i l^\mu) + C\theta^{i+1} \left(2\left(\frac{\epsilon_0}{2}\right)^{-3} \frac{D_0}{2^i}\right)^2. \tag{59}$$

This immediately implies

$$\sup_{B_{\theta^{i+1}}(0)} d(w, {}_{i+1}l) \leq C\theta^{1+\alpha} \sup_{B_{\frac{\theta^i}{2}}(0)} d(w, {}_i l) + mC\theta^{i+1} \left(2\left(\frac{\epsilon_0}{2}\right)^{-3} \frac{D_0}{2^i}\right)^2 \tag{60}$$

hence

$$\begin{aligned}
\sup_{B_{\theta^{i+1}}(0)} d(w, {}_{i+1}l) &\leq C\theta^{i+1}\theta^\alpha \frac{D_0}{2^{i-1}} + mC\theta^{i+1} \frac{D_0^2}{\epsilon_0^6 2^{2i-8}} \ \text{ (by (58))} \\
&< \theta^{i+1} \frac{D_0}{2^{i+4}} \ \text{ (by (49) and (52)).}
\end{aligned}$$

Combined with (57), we obtain

$$\begin{aligned}
\sup_{B_{\theta^{i+1}}(0)} d(v, {}_{i+1}l) &\leq \sup_{B_{\theta^{i+1}}(0)} d(v, w) + \sup_{B_{\theta^{i+1}}(0)} d(w, {}_{i+1}l) \\
&< \theta^{i+1} \frac{D_0}{2^{i+3}}.
\end{aligned} \tag{61}$$

This implies the first inequality of (45). Furthermore, note that ${}_{i+1}l^\mu_\varphi = w^\mu_\varphi$ by definition (cf. Theorem 16). Since $\theta \in (0, \frac{1}{8})$,

$$|{}_{i+1}l^\mu_\varphi(x)| = |w^\mu_\varphi(x)| \le 2\left(\frac{\theta^i\epsilon_0}{2}\right)^{-3}\frac{\theta^i D_0}{2^i} \le \left(\frac{\theta^{i+1}\epsilon_0}{2}\right)^{-3}\frac{\theta^{i+1}D_0}{2^{i+1}}. \qquad (62)$$

Thus, we conclude ${}_{i+1}l^\mu$ maps into $\overline{\mathbf{H}}[\left(\frac{\theta^{i+1}\epsilon_0}{2}\right)^{-3}\frac{\theta^{i+1}D_0}{2^{i+1}}, t^\mu_*]$,

We now proceed with the proof of the second inequality of (45). Let $A^\mu = 0$ for $\mu = m+1, \ldots, k-j$ for simplicity. Since ${}_il^\mu_\varrho \circ R^\mu(x)$ and $A^\mu x^1$ are both affine functions and $v^\mu(0) = P_0$, we have for every $x \in B_{\theta^i}(0)$

$$\begin{aligned}
&|{}_il^\mu_\varrho \circ R^\mu(\theta x) - A^\mu\theta x^1| \\
&= |(1-\theta){}_il^\mu_\varrho(0) + \theta({}_il^\mu_\varrho \circ R^\mu(x) - A^\mu x^1)| \\
&\le (1-\theta)d({}_il^\mu_\varrho \circ R^\mu(0), v^\mu(0)) + \theta|{}_il^\mu_\varrho \circ R^\mu(x) - A^\mu x^1| \\
&\le (1-\theta)d({}_il^\mu_\varrho \circ R^\mu(0), v^\mu(0)) + \theta d({}_il^\mu \circ R^\mu(x), v^\mu \circ R^\mu(x)) \\
&+ \theta|v^\mu_\varrho \circ R^\mu(x) - A^\mu x^1| \quad \text{(by Lemma 14)} \\
&< \theta^i\frac{D_0}{2^i} + \theta^{i+1}{}_i\delta \quad \text{(by (44))} \\
&= \theta^{i+1}\left({}_i\delta + \frac{D_0\theta^{-1}}{2^i}\right)
\end{aligned}$$

which implies

$$\sup_{B_{\theta^{i+1}}(0)} |{}_il^\mu_\varrho \circ R^\mu(x) - A^\mu x^1| \le \theta^{i+1}\left({}_i\delta + \frac{D_0\theta^{-1}}{2^i}\right). \qquad (63)$$

Thus, for $x \in B_{\theta^{i+1}}(0)$

$$\begin{aligned}
&|v^\mu_\varrho \circ R^\mu(x) - A^\mu x^1| \\
&\le |v^\mu_\varrho \circ R^\mu(x) - {}_il^\mu_\varrho \circ R^\mu(x)| + |{}_il^\mu_\varrho \circ R^\mu(x) - A^\mu x^1| \\
&\le d_{\mathbf{H}}(v^\mu \circ R^\mu(x), {}_il^\mu \circ R^\mu(x)) + |{}_il^\mu_\varrho \circ R^\mu(x) - A^\mu x^1| \quad \text{(by Lemma 14)} \\
&< \theta^i\frac{D_0}{2^i} + \theta^{i+1}\left({}_i\delta + \frac{D_0\theta^{-1}}{2^i}\right) \quad \text{(by (44) and (63))} \\
&< \theta^{i+1}\left({}_i\delta + \frac{2D_0\theta^{-1}}{2^i}\right) \\
&< \theta^{i+1}\sum_{k=0}^{i+1}\frac{\theta^{-1}D_0}{2^{k-2}} \quad \text{(by (44))}.
\end{aligned}$$

This is the second inequality of (45).

Finally, we will prove the third inequality of (45). Since $l^\mu(x) = (A^\mu x^1, 0)$ and since by (44)

$$ {}_i\delta < \sum_{k=0}^{i} \frac{\theta^{-1} D_0}{2^{k-2}} \leq 8\theta^{-1} D_0, $$

we conclude from (63) that

$$ \sup_{B_{\theta^{i+1}}(0)} d(({}_il^\mu_\varrho \circ R^\mu(x), 0), l^\mu(x)) = \sup_{B_{\theta^{i+1}}(0)} |{}_il^\mu_\varrho \circ R^\mu(x) - A^\mu x^1| < 9\theta^i D_0. $$

Thus, for $x \in B_{\theta^{i+1}}(0)$,

$$ \begin{aligned} & d({}_il^\mu \circ R^\mu(x), ({}_il^\mu_\varrho \circ R^\mu(x), 0)) \\ \leq \;& ({}_il^\mu_\varrho \circ R^\mu(x))^3 |{}_il^\mu_\varphi \circ R^\mu(x)| \\ < \;& \theta^{3i} (A + 9D_0)^3 \cdot 2 \left(\frac{\theta^i \epsilon_0}{2} \right)^{-3} \frac{\theta^i D_0}{2^i} \quad \text{(by (44))} \\ \leq \;& \theta^i \frac{2^3 (A + 9D_0)^3}{\epsilon_0^3} \cdot D_0. \end{aligned} $$

Combining the above two inequalities, we obtain

$$ d({}_il^\mu \circ R^\mu(x), l^\mu(x)) < \theta^i \left(\frac{2^3 (A + 9D_0)^3}{\epsilon_0^3} + 9 \right) D_0. $$

Combined with (44),

$$ \begin{aligned} \sup_{B_{\theta^{i+1}}(0)} d(v^\mu, l^\mu) &\leq \sup_{B_{\theta^{i+1}}(0)} d(v^\mu, {}_il^\mu \circ R^\mu) + \sup_{B_{\theta^{i+1}}(0)} d({}_il^\mu \circ R^\mu, l^\mu) \\ &< \theta^i \left(\frac{2^3 (A + 9D_0)^3}{\epsilon_0^3} + 10 \right) D_0. \end{aligned} $$

Hence,

$$ \begin{aligned} \sup_{B_{\theta^{i+1}}(0)} d(v, l) &\leq \sup_{B_{\theta^{i+1}}(0)} \sum_{\mu=1}^{m} d({}_il^\mu \circ R^\mu(x), l^\mu(x)) \\ &< m\theta^i \left(\frac{2^3 (A + 9D_0)^3}{\epsilon_0^3} + 10 \right) D_0. \end{aligned} $$

Q.E.D.

Proposition 25 *Given $c_0 \geq 1$, $E_0, A^1, \ldots, A^m > 0$, there exist $D_0 \in (0,1)$ and $c > 0$ with the following property:*

Assume

- *The map*

$$l = (l^1 \circ (R^1)^{-1}, \ldots, l^m \circ (R^m)^{-1}, l^{m+1}, \ldots, l^{k-j}) : B_{\theta^i}(0) \to \overline{\mathbf{H}}^{k-j}$$

  is such that R^μ is a rotation,

  $l^\mu(x) = (A^\mu x^1, 0)$ in coordinates (ϱ, φ) anchored at $t_*^\mu \in (-\infty, \frac{3}{2})$ (cf. (15))

  and

$$d_{\mathbf{H}}(P_0, l^\mu(0)) = c_\rho(0, t_*^\mu) < \frac{1}{\sqrt{8}} \text{ (cf. (14))}$$

  for $\mu = 1, \ldots, m$ and

$$_i l^\mu \text{ is identically equal to } P_0$$

  for $\mu = m+1, \ldots, k-j$.

- *The map*

$$v = (v^1, \ldots, v^{k-j}) : (B_1(0), g) \to \overline{\mathbf{H}}^{k-j}$$

  is such that g is a normalized metric,

$$E^v(\vartheta) \leq \vartheta^n E_0, \qquad \sup_{B_{\frac{1}{2}}(0)} d(v, l) < D_0$$

  and

  for $\vartheta \in (0, \frac{1}{24})$, $R \in [\frac{5}{8}, \frac{7}{8}]$ and a harmonic map $w : (B_{\vartheta R}(0), g) \to \overline{\mathbf{H}}^{k-j}$,

$$\sup_{B_{\frac{15\vartheta R}{16}}(0)} d^2(v, w) \leq \frac{c_0}{(\vartheta R)^{n-1}} \int_{\partial B_{\vartheta R}(0)} d^2(v, w) d\Sigma + c\vartheta^3. \quad (64)$$

- *The metric g is sufficiently close to the Euclidean metric such that if we denote Vol and Vol_g to be the volume with respect to the standard Euclidean metric and the metric g respectively, then*

$$\frac{15}{16}Vol(S) \leq Vol_g(S) \leq \frac{17}{16}Vol(S)$$

  for any smooth submanifold S of $B_1(0)$.

Then $v(0) \neq \mathcal{P}_0$.

Proof. Let θ, ϵ_0 and D_0 be as in the Inductive Lemma 24 and let $c = \frac{\theta^2 D_0^2}{2^8}$. By assumption,

$$\sup_{B_{\frac{1}{2}}(0)} d(v,l) < D_0. \tag{65}$$

We will also assume $v(0) = \mathcal{P}_0$. In order to arrive at a contradiction, we will apply Inductive Lemma 24 starting with $l = {}_0l$ and ${}_0\delta = D_0$ (cf. assumption (44) of the Inductive Lemma 24). To do so, we need to verify assumption (39) of Inductive Lemma 24; in other words, we need to show

$$a[2\left(\frac{\theta^i\epsilon_0}{2}\right)^{-3}\frac{\theta^i D_0}{2^i}, t_*^\mu] = a[\frac{16D_0}{\epsilon_0^3\theta^{2i}2^i}, t_*^\mu] < \frac{\theta^i}{2}.$$

For this purpose, we note the constants θ and ϵ_0 are *chosen before* the constant D_0 in the proof of Inductive Lemma 24; hence, there is no loss of generality in assuming that D_0 is chosen sufficiently small (cf. (52)) such that

$$\frac{8D_0}{\epsilon_0^3} < 1 \tag{66}$$

and

$$m\left(\frac{2^3\left(A+9D_0\right)^3}{\epsilon_0^3} + 10\right)\theta^{-1}D_0 < \frac{1}{\sqrt{8}}. \tag{67}$$

For $\mu = 1, \ldots, m$, recall that t_*^μ is the address of l^μ (cf. (13)). Reordering if necessary, we can assume

$$t_*^1 = \max\{t_*^1, \ldots, t_*^m\}. \tag{68}$$

Let i_0 be the non-negative integer such that

$$\frac{\theta^{i_0+1}}{\sqrt{8}} \leq c_\rho(0, t_*^1) < \frac{\theta^{i_0}}{\sqrt{8}}. \tag{69}$$

Recall by (10) and (11) that $t \mapsto c_\rho(0,t) =: f(t)$ satisfies

$$f'(t) = f^3(t) \text{ with } f(1) = 1.$$

Solving this differential equation, we obtain

$$f(t) = \frac{1}{\sqrt{3-2t}}$$

and

$$t = \frac{3}{2} - \frac{1}{2f^2(t)}.$$

In particular, since $f(t_*^1) = c_\rho(0, t_*^1) < \frac{\theta^{i_0}}{\sqrt{8}}$, we have

$$-t_*^1 = -\frac{3}{2} + \frac{1}{2f^2(t_*)} > -\frac{3}{2} + \frac{4}{\theta^{2i_0}}. \tag{70}$$

Therefore, if

$$i \in \{1, 2, \ldots, i_0\} \quad \text{and} \quad |t_*^1 - t| \leq \frac{16D_0}{\epsilon_0^3 \theta^{2i} 2^i},$$

then by (66) and (70)

$$3 - 2t > 3 - 2t_*^1 - \frac{32D_0}{\epsilon_0^3 \theta^{2i} 2^i} > \frac{8}{\theta^{2i_0}} - \frac{4}{\theta^{2i} 2^i} \geq \frac{8}{\theta^{2i}} - \frac{4}{\theta^{2i}} = \frac{4}{\theta^{2i}}.$$

In turn, this implies

$$c_\rho(0,t) = f(t) = \frac{1}{\sqrt{3-2t}} < \frac{\theta^i}{2}.$$

In summary, we have shown

$$i \in \{1, 2, \ldots, i_0\} \quad \Rightarrow \quad a[\frac{16D_0}{\epsilon_0^3 \theta^{2i} 2^i}, t_*^1] = \max_{\{\varphi : |\varphi| \leq \frac{16D_0}{\epsilon_0^3 \theta^{2i} 2^i}\}} c_\rho(0, t_*^1 + \varphi) < \frac{\theta^i}{2}.$$

By (10), $t \mapsto c_\rho(0,t)$ is an increasing function. Since $t_*^1 \geq t_*^\mu$ for $\mu = 2, \ldots, m$, this implies that

$$i \in \{1, 2, \ldots, i_0\} \quad \Rightarrow \quad a[\frac{16D_0}{\epsilon_0^3 \theta^{2i} 2^i}, t_*^\mu] = \max_{\{\varphi : |\varphi| \leq \frac{16D_0}{\epsilon_0^3 \theta^{2i} 2^i}\}} c_\rho(0, t_*^\mu + \varphi) < \frac{\theta^i}{2}.$$

In other words, the assumption (39) of Inductive Lemma 24 is satisfied for $i = 0, 1, 2, \ldots, i_0$. We can now complete the proof by applying the Inductive Lemma 24 as follows:

Let ${}_0l = l$ and ${}_0\delta = D_0$ (cf. assumption (44) of the Inductive Lemma 24). By (65) and Lemma 14,

$$\begin{cases} \sup\limits_{B_\theta(0)} d(v, {}_0l) < D_0 \\ \sup\limits_{B_\theta(0)} |v_\varrho^\mu \circ R^\mu - A^\mu x^1| < {}_0\delta < 4\theta^{-1} D_0. \end{cases}$$

We apply the Inductive Lemma 24 for $i = 1, 2, \ldots, i_0$ to obtain

$$\sup_{B_{\theta^{i_0+1}}(0)} d(v, l) < m\theta^{i_0+1} \left(\frac{2^3 \left(A + 9D_0\right)^3}{\epsilon_0^3} + 10 \right) \theta^{-1} D_0. \tag{71}$$

Thus,

$$\begin{aligned} \frac{\theta^{i_0+1}}{\sqrt{8}} &\leq c_\rho(0, t_*^1) \qquad \text{(by (68) and (69))} \\ &= d_{\mathbf{H}}(P_0, l^1 \circ (R^1)^{-1}(0)) \qquad \text{(by (14))} \\ &= d_{\mathbf{H}}(v^1(0), l^1 \circ (R^1)^{-1}(0)) \text{ (by the assumption that } v(0) = \mathcal{P}_0) \\ &\leq d(v(0), l(0)) \\ &\leq m\theta^{i_0+1} \left(\frac{2^3 \left(A + 9D_0\right)^3}{\epsilon_0^3} + 10 \right) \theta^{-1} D_0 \qquad \text{(by (71))} \\ &< \frac{\theta^{i_0+1}}{\sqrt{8}} \qquad \text{(by (67)).} \end{aligned}$$

This contradicts our assumption that $v(0) = \mathcal{P}_0$. Q.E.D.

We now come to the main result regarding asymptotic harmonic maps.

Proposition 26 *There exists $\epsilon_0 > 0$ such that if a sequence of asymptotic harmonic maps $\{v_i : (B_1(0) \subset \mathbf{R}^n, g_i) \to \overline{\mathbf{H}}^{k-j}\}$ with $v_i(0) = \mathcal{P}_0$ converges locally uniformly in the pullback sense (cf. [KS2] Definition 3.3) to a non-constant homogeneous harmonic map $v_0 : (B_1(0), g_0) \to (Y_0, d_0)$ into an NPC space, then*

$$Ord^{v_0}(0) \geq 1 + \epsilon_0 \quad \textit{and} \quad \dim_{\mathcal{H}}(\mathcal{S}_0(v_0)) \leq n-2.$$

PROOF. Let $w_i : B_{\frac{3}{4}}(0) \to \overline{\mathbf{H}}^{k-j}$ be the harmonic map whose boundary values agree with that of $v_i|_{B_{\frac{3}{4}}(0)}$. Letting

$$R = \vartheta = \frac{3}{4}, \; r = \frac{2}{3} \tag{72}$$

and $w = w_i$ in (iv) of Definition 22, we obtain

$$\lim_{i\to\infty} \sup_{B_{\frac{1}{2}}(0)} d^2(v_i, w_i) = 0. \tag{73}$$

The total energies of $\{w_i\}$ are uniformly bounded since the total energies of $\{v_i\}$ in $B_{\frac{3}{4}}(0)$ are uniformly bounded by (ii) of Definition 22. Thus, $\{w_i^\mu\}$ have uniform local Lipschitz estimates. This then implies, by Proposition 3.7 of [KS2], that a subsequence of $\{w_i^\mu\}$ (for all $\mu = 1, \ldots, k-j$) converges locally uniformly in the pullback sense to a limit map by $v_0^\mu : B_{\frac{3}{4}}(0) \to Y_0^\mu$. By (73), the sequence $\{w_i\}$ also converges locally uniformly in the pullback sense to $v_0|_{B_{\frac{1}{2}}(0)}$ and $d(w_i(0), \mathcal{P}_0) \to 0$. Thus,

$$v_0|_{B_{\frac{3}{4}}(0)} = (v_0^1, \ldots, v_0^{k-j}) : B_{\frac{3}{4}}(0) \to Y_0 = Y_0^1 \times \ldots \times Y_0^{k-j}.$$

Since the Lipschitz constants of $\{w_i^\mu\}$ is uniformly bounded in $B_{\frac{1}{2}}(0)$, $v_0^\mu|_{B_{\frac{1}{2}}(0)}$ is a harmonic map by Theorem 3.11 of [KS2]. Since $v_0|_{B_{\frac{1}{2}}(0)}$ is a homogeneous map, so is each of $v_0^1, \ldots, v_0^{k-j}$. By reordering if necessary, we can assume $v_0^1, \ldots, v_0^m$ are non-constant maps and $v_0^{m+1}, \ldots, v_0^{k-j}$ identically equal to P_0.

By Lemma 12, it suffices to prove that $Ord^{v_0}(0) \neq 1$. Thus, with the intent of arriving at a contradiction, we assume $Ord^{v_0}(0) = 1$ which implies that $Ord^{v_0^\mu}(0) = 1$ for $\mu = 1, \ldots, m$. By Lemma 13, we conclude that, for

each $\mu \in \{1, \dots, m\}$, there exists a sequence of translation isometries $\{T_i^\mu\}$, a rotation $R^\mu : \mathbf{R}^n \to \mathbf{R}^n$ and a sequence of symmetric homogeneous degree 1 maps $\{l_i^\mu\}$ with $d_{\mathbf{H}}(P_0, l_i^\mu(0)) \to 0$ and stretch converging, say to A^μ, such that

$$\lim_{i\to\infty} \sup_{B_{\frac{1}{2}}(0)} d(w_i^\mu, T_i^\mu \circ l_i^\mu \circ R^\mu) = 0. \tag{74}$$

The above defines the constants $A^1, \dots, A^m$. Combined with (73), we see that

$$\lim_{i\to\infty} \sup_{B_{\frac{1}{2}}(0)} d(v_i^\mu, T_i^\mu \circ l_i^\mu \circ R^\mu) = 0.$$

With the constants as in (72), let $c_0 \geq 1$ be as in Definition 22, condition (iv) and $E_0 > 0$ the Lipschitz bound of $\{v_i\}$ in $B_{\frac{1}{2}}(0)$. Let $D_0 > 0$ and $c \geq 0$ be as in Proposition 25. By (iv) of Definition 22, we can fix i sufficiently large such that $c_i \leq c$ and by (74),

$$\sup_{B_{\frac{1}{2}}(0)} d(v_i^\mu, T_i^\mu \circ l_i^\mu \circ R^\mu) < \frac{D_0}{m}, \quad \forall \mu \in \{1, \dots, m\}$$

and

$$d(T_i^\mu \circ l_i^\mu \circ R^\mu(0), P_0) < \frac{1}{\sqrt{8}}, \quad \forall \mu \in \{1, \dots, m\}.$$

Define $l = (l^1, \dots, l^{k-j}) : B_1(0) \to \overline{\mathbf{H}}^{k-j}$ by setting

$$l^\mu = T_i^\mu \circ l_i^\mu \circ R^\mu, \ \forall \mu = 1, \dots, m \ \text{ and } \ l^\mu \equiv P_0, \ \forall \mu = m+1, \dots, k-j.$$

Thus,

$$\sup_{B_{\frac{1}{2}}(0)} d(v_i, l) < D_0.$$

Thus, we can apply Proposition 25 to conclude that $v_i(0) \neq \mathcal{P}_0$. This contradiction proves $Ord^{v_0}(0) \neq 1$. Q.E.D.

Corollary 27 *Let $v : B_{\sigma_\star}(x_\star) \to \overline{\mathbf{H}}^{k-j}$ satisfy properties (P1) and (P2) with respect to $S \subset v^{-1}(\mathcal{P}_0)$. If, for each $x \in S$, there exists a sequence of blow up maps of v at x that is a sequence of asymptotically harmonic maps, then*

$$\dim_{\mathcal{H}}(S) \leq n-2.$$

PROOF. Proposition 26 and (4) imply that if $x \in S$ and v_0 a tangent map of v at x_0, then $ord^v(x) = ord^{v_0}(x) \geq 1 + \epsilon_0$. The Corollary follows immediately from Theorem 10. Q.E.D.

Corollary 28 *Let $u : B_{\sigma_\star}(x_\star) \rightarrow \overline{\mathbf{H}}$ be a harmonic map.*

$$\dim_{\mathcal{H}}(\mathcal{S}(u)) \leq n - 2.$$

PROOF. Follows immediately from Corollary 27 and Remark 23. Q.E.D.

4 Blow up maps of u and higher order points

The goal of this section is to show that the set of singular points of order ≥ 1 is of Hausdorff codimension 1. As in (28), let

$$u = (V, v) : (B_{\sigma_\star}(x_\star), g) \rightarrow (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G)$$

be a local representation of a harmonic map into $\overline{\mathcal{T}}$. For $x_0 \in \hat{\mathcal{S}}_j(u)$, identify $x_0 = 0$ via normal coordinates for the metric g and identify $V(x_0) = 0$ via normal coordinates for the metric H. In this section, we consider the family of blow up maps $\{u_\sigma\}$ described in Remark 8; in other words, the scaling factor is given by

$$\mu(\sigma) = \sqrt{\frac{I^u(\sigma)}{\sigma^{n-1}}}. \tag{75}$$

We now define the maps

$$V_\sigma : (B_1(0), g_\sigma) \rightarrow (\mathbf{C}^j, H_{\mu(\sigma)}) \text{ and } v_\sigma : (B_1(0), g_\sigma) \rightarrow (\mathbf{H}^{k-j}, h)$$

by setting

$$V_\sigma(x) = \mu^{-1}(\sigma) V(\sigma x) \text{ and } v_\sigma(x) = \mu^{-1}(\sigma) v(\sigma x)$$

and

$$g_\sigma(x) = g(\sigma x) \text{ and } H_{\mu(\sigma)}(y) = H(\mu(\sigma) y).$$

Thus, the blow up map of u at $x_0 = 0$ can be written as

$$u_\sigma = (V_\sigma, v_\sigma) : (B_1(0), g_\sigma) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_{G_{\mu(\sigma)}}) \tag{76}$$

where $d_{G_{\mu(\sigma)}}$ is the distance function induced by the metric

$$G_{\mu(\sigma)}(y, P) = G(\mu(\sigma)y, \mu(\sigma)P).$$

Lemma 29 *(i) There exists a constant $C > 0$ such that for $P, Q \in \mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}$ at distance at most λ from P_0,*

$$\left(1 - C\lambda^2\right) \leq \frac{d_{H\oplus h}(P,Q)}{d_G(P,G)} \leq \left(1 + C\lambda^2\right).$$

(ii) If $h = (W, w) : B_1(0) \to \mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}$ is Lipschitz continuous in $B_R(0)$, for some $R \in (0,1)$, then there exists $C > 0$ such that

$$\left| |\nabla h|^2(x) - \left(|\nabla W|^2(x) + |\nabla w|^2(x)\right)\right| \leq C d^2(w(x), P_0)$$

for almost every $x \in B_R(0)$ and every $x \in \mathcal{R}(u) \cap B_R(0)$.
(iii) Given $R \in (0,1)$, there exists $C > 0$ such that for almost every $x \in B_R(0)$, every $x \in \mathcal{R}(u) \cap B_R(0)$ and $\sigma > 0$ sufficiently small, the blow up map

$$u_\sigma = (V_\sigma, v_\sigma) : (B_1(0), g_\sigma) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_{G_{\mu(\sigma)}})$$

of the harmonic map u with scaling factor (75) satisfies

$$(1 + C\sigma^2)^{-1}|\nabla u_\sigma|^2(x) \leq |\nabla V_\sigma|^2(x) + \sum_{i=1}^{k-j} |\nabla v_\sigma^i|^2(x) \leq (1 + C\sigma^2)|\nabla u_\sigma|^2(x).$$

PROOF. Part (i) follows from the C^0-estimates of G contained in (30) and the same argument as [DM1] inequality (36). The inequalities of (ii) hold for almost every $x \in B_R(0)$ by the definition of energy density (cf. [KS1]) and (i) and by smoothness they also hold for every $x \in \mathcal{R}(u) \cap B_R(0)$. Finally, since $\{u_\sigma\}$ is a set of harmonic maps whose total energy is bounded independently of σ, they are uniformly Lipschitz continuous in $B_R(0)$. Thus, assertion (iii) follows from (ii). Q.E.D.

Lemma 30 *Let $u : \Omega \to \overline{\mathcal{T}}$, $x_\star \in \Omega$ and $u = (V, v)$ as in (28). There exists a sequence $\sigma_i \to 0$ such that the blow up maps $\{u_{\sigma_i} = (V_{\sigma_i}, v_{\sigma_i})\}$ at $x_\star$ converge locally uniformly in the pullback sense to a nonconstant map*

$$u_* = (V_*, v_*^1, \ldots, v_*^{k-j}) : B_1(0) \to \mathbf{C}^j \times Y_{1*} \times \ldots \times Y_{k-j*}$$

with $(V_*, v_*^1, \ldots, v_*^{k-j})$ a homogeneous degree α harmonic map. Furthermore, $V_*, v_*^1, \ldots, v_*^{k-j}$ are homogeneous of degree α and V_{σ_i}, $v_{\sigma_i}^m$ converge to V_*, v_*^m respectively.

PROOF. By definition, the maps $\{u_\sigma\}$ are normalized such that $I^{u_\sigma}(1) = 1$. Since

$$Ord^u(x_\star) = \lim_{\sigma \to 0} \frac{\sigma E^u(\sigma)}{I^u(\sigma)} = \lim_{\sigma \to 0} \frac{E^{u_\sigma}(1)}{I^{u_\sigma}(1)} = \lim_{\sigma \to 0} E^{u_\sigma}(1),$$

we have that $E^{u_\sigma}(1) \leq 2Ord^u(x_\star)$ for sufficiently small $\sigma > 0$. By [KS2] Theorem 2.4.6, u_σ has a local Lipschitz bound uniformly for sufficiently small $\sigma > 0$. Combined with Lemma 29, we conclude that for any compactly contained subset K of $B_1(0)$, there exists $C > 0$ such that

$$|\nabla V_\sigma|^2, \ |\nabla v_\sigma^1|^2, \ldots, |\nabla v_\sigma^{k-j}|^2 \leq C \tag{77}$$

in K for sufficiently small σ (with respect to the metric $g(0)$ on the domain which is uniformly equivalent to g_σ for σ small).

Let $\sigma_i \to 0$ be such that u_{σ_i} converges to a tangent map u_* locally uniformly in the pullback sense. (We refer to [DM1] Section 2 for more details on the construction of a tangent map.) Additionally, [KS2] Proposition 3.7 and a diagonalization argument imply that there exist a subsequence (which for the sake of simplicity we call again $\sigma_i \to 0$), NPC spaces $(Y_{1*}, d_{1*}), \ldots, (Y_{k-j*}, d_{k-j*})$ and maps $V_* : \mathbf{R}^n \to (\mathbf{C}^j, H(0))$, $v_*^1 : \mathbf{R}^n \to (Y_{1*}, d_{1*}), \ldots, v_*^{k-j} : \mathbf{R}^n \to (Y_{k-j*}, d_{k-j*})$ such that $V_{\sigma_i}, v_{\sigma_i}^1, \ldots, v_{\sigma_i}^{k-j}$ converge locally uniformly in the pull-back sense to $V_*, v_*^1, \ldots, v_*^{k-j}$ respectively. Furthermore, Lemma 29 implies that for $x', x'' \in B_1(0)$,

$$d^2_{G_{\sigma_i}}(u_{\sigma_i}(x'), u_{\sigma_i}(x'')) = d^2_{H_{\sigma_i}}(V_{\sigma_i}(x'), V_{\sigma_i}(x'')) + \sum_{\mu=1}^{k-j} d^2_{\mathbf{H}}(v^\mu_{\sigma_i}(x'), v^\mu_{\sigma_i}(x'')) + O(\sigma_i^2).$$

Thus, we conclude that u_{σ_i} converges locally uniformly in the pullback sense to

$$(V_*, v_*^1, \ldots, v_*^{k-j}) : B_1(0) \to \mathbf{C}^j \times Y_{1*} \times \ldots \times Y_{k-j*}$$

and

$$d_*^2(u_*(x'), u_*(x'')) = |V_*(x') - V_*(x'')|^2 + \sum_{m=1}^{k-j} d_{m*}^2(v_*^m(x'), v_*^m(x'')).$$

The map $(V_*, v_*^1, \ldots, v_*^{k-j})$ is harmonic by [KS2] Theorem 3.11. Furthermore, the homogeneity of tangent map u_* implies the homogeneity of V_* and v_*^m. Q.E.D.

The following is the main result of this section.

Proposition 31 *If $u : \Omega \to \overline{\mathcal{T}}$ is a harmonic map from an Riemannian domain, then the set of higher order points is of Hausdorff co-dimension 2; i.e. if Ω is an n-dimensional domain, then*

$$\dim_{\mathcal{H}}(\mathcal{S}_0(u)) \leq n - 2.$$

PROOF. By Corollary 11, it suffices to show that there exists $\epsilon_0 > 0$ such that at every $x_\star \in \Omega$ and tangent map u_* of u at $x_\star$,

$$Ord^{u_*}(0) = 1 \quad \text{or} \quad Ord^{u_*}(0) \geq 1 + \epsilon_0 \tag{78}$$

and

$$\dim_{\mathcal{H}}(\mathcal{S}_0(u_*)) \leq n - 2. \tag{79}$$

For $x_\star \in \mathcal{R}(u)$, statements (78) and (79) obviously hold (with $\epsilon_0 = 1$) since all the strata of $\overline{\mathcal{T}}$ are smooth manifolds. Thus, now consider $x_\star \in \hat{\mathcal{S}}_j(u)$. By Lemma 30, there exists a sequence of blow $\{u_{\sigma_i} = (V_{\sigma_i}, v_{\sigma_i})\}$ at $x_\star$ that converges locally uniformly in the pullback sense to a map

$$u_* = (V_*, v_*^1, \ldots, v_*^{k-j}) : B_1(0) \to \mathbf{C}^j \times Y_{1*} \times \ldots \times Y_{k-j*}$$

with $V_*, v_*^1, \ldots, v_*^{k-j}$ homogeneous harmonic maps and V_{σ_i}, $v_{\sigma_i} = (v_{\sigma_i}^1, \ldots, v_{\sigma_i}^{k-j})$ converging locally uniformly in the pullback sense to V_*, $v_* = (v_*^1, \ldots, v_*^{k-j})$ respectively. First, assume V_* is non-constant. Then $Ord^{u_*}(0) = Ord^{V_*}(0)$, and since V_* is a harmonic map into Euclidean space, statements (78) and (79) obviously hold (again with $\epsilon_0 = 1$). Alternatively, assume that V_* is a constant map. In this case,

$$\lim_{\sigma_i \to 0} \sup_{\partial B_r(0)} d(V_{\sigma_i}(0), V_{\sigma_i}) = 0, \quad \forall r \in (0, 1). \tag{80}$$

Define

$$\hat{u}_{\sigma_i} : B_{\frac{1}{2}}(0) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_{G_{\mu(\sigma)}}), \quad \hat{u}_{\sigma_i} = (V_{\sigma_i}(0), v_{\sigma_i})$$

and let

$$h_{\sigma_i} : B_{\frac{1}{2}}(0) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_{G_{\mu(\sigma)}}), \quad h_{\sigma_i} = (W_{\sigma_i}, w_{\sigma_i})$$

be the harmonic map with boundary values equal to $\hat{u}_{\sigma_i}$. Since h_{σ_i} and u_{σ_i} are harmonic maps, $d^2(h_{\sigma_i}, u_{\sigma_i},)$ is a weakly subharmonic function by [KS1] Lemma 2.4.2. Thus, noting that $v_{\sigma_i} = w_{\sigma_i}$ and $W_{\sigma_i} = V_{\sigma_i}(0)$ on $B_{\frac{1}{2}}(0)$,

$$\begin{aligned}
\lim_{\sigma_i \to 0} \sup_{B_{\frac{1}{4}}(0)} d^2(w_{\sigma_i}(x), v_{\sigma_i}(x)) &\leq C \lim_{\sigma_i \to 0} d^2(h_{\sigma_i}(x), u_{\sigma_i}(x)) \quad \text{(by Lemma 29)} \\
&\leq C \lim_{\sigma_i \to 0} \int_{\partial B_{\frac{1}{2}}(0)} d^2(h_{\sigma_i}, u_{\sigma_i}) d\Sigma \\
&\leq C \lim_{\sigma_i \to 0} \int_{\partial B_{\frac{1}{2}}(0)} d^2(V_{\sigma_i}(0), V_{\sigma_i}) d\Sigma \\
&= 0 \quad \text{(by (80))}.
\end{aligned}$$

Thus, the sequence $\{w_{\sigma_i}\}$ converges locally uniformly in the pullback sense to v_* and $w_{\sigma_i}(0) \to \mathcal{P}_0$. Applying Lemma 12 to a component map of w_{σ_i}, we that conclude there exists $\epsilon_0 \in (0, 1]$ satisfying (78) and (79). Q.E.D.

5 Proof of Theorem 1 and Theorem 2

In this section, we prove Theorems 1 and 2 by applying an inductive argument given in [DM1] with [DM2] and [DM3] being the key ingredients. To proceed, we need the following statements for a harmonic map $u : \Omega \to \overline{\mathcal{T}}$.

STATEMENT 1[j]: For any $x_\star \in \mathcal{S}_j(u)$ and a local representation $u = (V, v) : (B_{\sigma_\star}(x_\star), g) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G)$ as in (28), we have

$$\dim_{\mathcal{H}} \left(\mathcal{S}(u) \cap B_{\frac{\sigma_\star}{2}}(x_\star) \right) \leq n - 2.$$

STATEMENT 2[j]: For $x_\star \in \mathcal{S}_j(u)$, a local representation $u = (V, v)$:

$(B_{\sigma_\star}(x_\star), g) \to (\mathbf{C}^j \times \overline{\mathbf{H}}^{k-j}, d_G)$ as in (28) and any subdomain Ω compactly contained in

$$B_{\frac{\sigma_\star}{2}}(x_\star) \backslash \left(\mathcal{S}(u) \cap v^{-1}(P_0) \right),$$

there exists a sequence of smooth functions $\{\psi_i\}$ with $\psi_i \equiv 0$ in a neighborhood of $\mathcal{S}(u) \cap \overline{\Omega}$, $0 \leq \psi_i \leq 1$, $\psi_i \to 1$ for all $x \in \Omega \backslash \mathcal{S}(u)$ such that

$$\lim_{i \to \infty} \int_{B_{\frac{\sigma_\star}{2}}(x_\star)} |\nabla \nabla u| |\nabla \psi_i| \; d\mu = 0.$$

We will prove STATEMENT $1[j]$ and STATEMENT $2[j]$ for all $j \in \{1, \ldots, k\}$ by a backwards induction on j. The initial step of the induction is for $j = k$. Since $\hat{\mathcal{S}}_k(u) = \emptyset$, Proposition 31 immediately implies STATEMENT $1[k]$ and STATEMENT $2[k]$. The inductive step is to prove that STATEMENT $1[j]$ and STATEMENT $2[j]$ hold under:

> **Inductive Assumption.** STATEMENT $1[m]$ and STATEMENT $2[m]$ hold for $m = j+1, j+2, \ldots, k$.

To proceed, we need to check that Assumptions 1 - 6 of [DM1] are satisfied.

Lemma 32 (Assumption 1) *The metric space $(\overline{\mathbf{H}}^{k-j}, d_h)$ is an NPC space with a homogeneous structure with respect to $\mathcal{P}_0 = (P_0, \ldots, P_0) \in \overline{\mathbf{H}}^{k-j}$.*

PROOF. Indeed, using the homogeneous structure on $\overline{\mathbf{H}}$ defined by (5), we can define a continuous map $\mathbf{R}_{>0} \times \overline{\mathbf{H}}^{k-j} \to \overline{\mathbf{H}}^{k-j}$ by setting

$$(\lambda, (P^1, \ldots, P^{k-j})) \to (\lambda P^1, \ldots, \lambda P^{k-j}).$$

Q.E.D.

Lemma 33 (Assumption 2) *The metrics G, H and h satisfy estimates (30), (31), (32), (33) and (34) of Assumption 2.*

PROOF. This is proven in [DM3]. Q.E.D.

Lemma 34 (Assumption 3) *Let $u = (V, v)$ as in (28). The set $\mathcal{S}_j(u)$ satisfies the following:*

(i) $v(x) = \mathcal{P}_0$ *for* $x \in \mathcal{S}_j(u) \cap B_{\sigma_\star}(x_\star)$

(ii) $\dim_{\mathcal{H}}((\mathcal{S}(u)\backslash\mathcal{S}_j(u)) \cap B_{\frac{\sigma_\star}{2}}(x_\star)) \leq n-2.$

Proof. The inductive assumption along with Proposition 31 implies the assertion. Q.E.D.

Lemma 35 (Assumption 4) *Let* $u = (V, v)$ *as in (28). For* $B_R(x_0) \subset B_{\frac{\sigma_\star}{2}}(x_\star)$ *and any harmonic map* $w : (B_R(x_0), g) \to \overline{\mathbf{H}}^{k-j}$*, the set* $\mathcal{R}(u, w)$ *is of full measure in* $\mathcal{R}(u) \cap B_R(x_0)$*. Here, recall that* $\mathcal{R}(u, w)$ *is the set of points* $x \in \mathcal{R}(u) \cap B_R(x_0)$ *with the property that there exists* $r > 0$ *such that* $v(B_r(x)), w(B_r(x))$ *map into the same stratum of* $\overline{\mathbf{H}}^{k-j}$*.*

Proof. From Corollary 28, we have $\dim_{\mathcal{H}}(\mathcal{S}(w)) \leq n-2$. Thus, $\mathcal{R}(w)$ is of full measure in $B_\sigma(x_0)$ which immediately implies $\mathcal{R}(u, w)$ is of full measure in $\mathcal{R}(u) \cap B_\sigma(x_0)$. Q.E.D.

Before we move on to Assumption 5, we need the following preliminary lemmas.

Lemma 36 *Let* $u = (V, v)$ *as in (28). Then there exists a sequence* $u_{\sigma_i} = (V_{\sigma_i}, v_{\sigma_i})$ *of blow up maps of* u *at* $x_\star$ *such that* v_{σ_i} *is a sequence of asymptotically harmonic maps.*

Proof. By Lemma 30, there exists a sequence of blow up maps $\{u_{\sigma_i} = (V_{\sigma_i}, v_{\sigma_i})\}$ that converges locally uniformly in the pullback sense to a map

$$u_* = (V_*, v_*^1, \ldots, v_*^{k-j}) : B_1(0) \to \mathbf{C}^j \times Y_{1*} \times \ldots \times Y_{k-j*}$$

with $V_*, v_*^1, \ldots, v_*^{k-j}$ homogeneous degree α harmonic maps and V_{σ_i} and v_{σ_i} converging to V_* and $v_* = (v_*^1, \ldots, v_*^{k-j})$ respectively. For the sequence $\{v_i = v_{\sigma_i}\}$, Property (i) of Definition 22 follows immediately from the definition of blow ups. Property (ii) follows from the fact that u_{σ_i} and hence v_{σ_i} is uniformly locally Lipschitz continuous. Since v_{σ_i} converges to v_*, we have Property (iii). Finally, we will prove Property (iv) follows from Lemma 20. Q.E.D.

Lemma 37 *Let $u : \Omega \to \overline{\mathcal{T}}$ be a harmonic map, $x_\star \in \mathcal{S}_j(u)$ and $u = (V, v)$ as in (28). There exists a sequence $u_{\sigma_i} = (V_{\sigma_i}, v_{\sigma_i})$ of blow up maps of u at $x_\star$ such that v_{σ_i} converges to a constant map and u_{σ_i}, V_{σ_i} converge to a tangent map u_* of u at $x_\star$.*

PROOF. By Lemma 30, there exists a sequence $u_{\sigma_i} = (V_{\sigma_i}, v_{\sigma_i})$ of blow up maps of u at $x_\star$ converging to a tangent map $u_* = (V_*, v_*)$ of u at $x_\star$. By assumption that $x_\star \in \mathcal{S}_j(u)$, we have $Ord^{u_*}(0) = 1$. By Lemma 36, $\{v_{\sigma_i}\}$ is a sequence of asymptotic harmonic maps. By Lemma 30, $\{v_{\sigma_i}\}$ converges locally uniformly to

$$(v_*^1, \ldots, v_*^{k-j}) : B_1(0) \to Y_*^1 \times \ldots \times Y_*^{k-j}.$$

such that $v_*^1, \ldots, v_*^{k-j}$ are homogeneous harmonic maps into an NPC space. Thus, we can apply Proposition 26 to conclude that v_* is identically constant. Q.E.D.

Lemma 38 (Assumption 5) *If $u = (V, v)$ as in (28), then*

$$|\nabla v|^2(x) = 0 \quad \textit{and} \quad |\nabla V|^2(x) = |\nabla u|^2(x) \textit{ for a.e. } x \in \mathcal{S}_j(u) \cap B_{\sigma_\star}(x_\star).$$

PROOF. Let $x \in \mathcal{S}_j(u) \cap B_{\sigma_\star}(x_\star)$ and identify $x = 0$ via normal coordinates. By Lemma 37, we can fix a sequence $\{u_{\sigma_i} = (V_{\sigma_i}, v_{\sigma_i})\}$ of blow up maps of u such that $\{u_{\sigma_i}\}$ and $\{V_{\sigma_i}\}$ converge to a tangent map $u_* = V_* : B_1(0) \to \mathbf{C}^j$ and v_{σ_i} converges to a constant map. Lemma 29 implies

$$E^{u_{\sigma_i}}(r) = \Big(E^{V_{\sigma_i}}(r) + E^{v_{\sigma_i}}(r)\Big) + O(\sigma_i^2). \tag{81}$$

Therefore,

$$\begin{aligned}
\limsup_{i\to\infty} E^{V_{\sigma_i}}(r) &\leq \limsup_{i\to\infty} (E^{V_{\sigma_i}}(r) + E^{v_{\sigma_i}}(r)) \\
&= \lim_{i\to\infty} E^{u_{\sigma_i}}(r) \quad \text{(by (81))} \\
&= E^{u_*}(r) \quad \text{(by [KS2] Theorem 3.11)} \\
&= E^{V_*}(r) \quad \text{(since } u_* = V_*) \\
&\leq \liminf_{i\to\infty} E^{V_{\sigma_i}}(r) \quad \text{(by [KS2] Theorem 3.8).}
\end{aligned}$$

This immediately implies

$$\lim_{i\to\infty} E^{V_{\sigma_i}}(r) = \lim_{i\to\infty} E^{u_{\sigma_i}}(r) \ \text{ and } \ \lim_{i\to\infty} E^{v_{\sigma_i}}(r) = 0. \tag{82}$$

Since $|\nabla v|^2$ is an integrable function, almost every point of $B_{\sigma_\star}(x_\star)$ is a Lebesgue point. In particular, at almost every $x \in \mathcal{S}_j(u) \cap B_{\sigma_\star}(x_\star)$,

$$\begin{aligned}
|\nabla v|^2(x) &= \lim_{i\to\infty} \frac{1}{Vol(B_{\sigma_i r}(x))} \int_{B_{\sigma_i r}(0)} |\nabla v|^2 d\mu \\
&= \lim_{i\to\infty} \frac{\mu_{\sigma_i}^2}{Vol(B_r(0))} \int_{B_r(0)} |\nabla v_{\sigma_i}|^2 d\mu_{\sigma_i} \\
&\leq \lim_{i\to\infty} \frac{C^2}{Vol(B_r(0))} \int_{B_r(0)} |\nabla v_{\sigma_i}|^2 d\mu_{\sigma_i} \\
&= 0 \quad \text{(by (82))}.
\end{aligned}$$

This implies the first assertion. The second follows immediately from the first. Q.E.D.

Finally, note that **Assumption 6** immediately follows from the inductive assumptions Statement 2$[j+1]$, $\ldots$, Statement 2$[k]$.

In summary, Assumptions 1-6 of Section 2.5 are satisfied. By Theorem 21, the singular component map satisfies v satisfies properties (P1) and (P2) with respect to $\mathcal{S}_j(u)$ as in Definition 9 and the monotonicity properties of (37) hold. Combined with Lemma 20, this implies the sequence of blow up maps $\{v_i = v_{\sigma_i}\}$ (from property (P2)) is a sequence of asymptotically harmonic maps. Combining this with Corollary 27, we obtain $\dim_{\mathcal{H}}(\mathcal{S}_j(u)) \leq n-2$. Now Statement 1$[j]$ follows immediately. Additionally, Statement 2$[j]$ follows from repeating the argument of [DM1] Section 11. Thus, induction completes the proof of Theorem 1 and Theorem 2.

6 Two dimensional domains

In this section, we prove Theorem 5, the regularity of harmonic maps from two dimensional domains. We first need the following preliminary lemma.

Lemma 39 *Let $u : \Omega \to \overline{\mathcal{T}}$ be a harmonic map from an n-dimensional Lipschitz Riemannian domain, Σ a connected submanifold of Ω (possibly $\Sigma =$*

Ω) and $\mathcal{T}'$ a stratum of $\overline{\mathcal{T}}$ (possibly $\mathcal{T}' = \mathcal{T}$). If $u(\Sigma) \cap \mathcal{T}' \neq \emptyset$ and $\Sigma \subset \mathcal{R}(u)$, then $u(\Sigma) \subset \mathcal{T}'$. Moreover, there exists a stratum $\mathcal{T}'$ of $\overline{\mathcal{T}}$ such that $u(\mathcal{R}(u)) \subset \mathcal{T}'$.

PROOF. Since $u(\Sigma) \cap \mathcal{T}' \neq \emptyset$, we have that $W := u^{-1}(\mathcal{T}') \cap \Sigma$ is a nonempty open subset of Σ. Assume on the contrary that $u(\Sigma) \not\subset \mathcal{T}'$, and let $x \in \partial W \cap \Sigma$. Since $\Sigma \subset \mathcal{R}(u)$, there exists $r > 0$ such that $u(B_r(x))$ is contained in a single stratum. Since $B_r(x) \cap W \neq \emptyset$, we conclude that $u(B_r(x)) \subset \mathcal{T}'$ contradicting the fact that $x \in \partial W \cap \Sigma$. This proves the first assertion. Since $\mathcal{S}(u)$ is of Hausdorff codimension 2 the set $\mathcal{R}(u)$ is connected (follows easily from [Schi] Corollary 4). Thus, the second assertion follows from the first. Q.E.D.

PROOF OF THEOREM 5. Using the fact that the dimension is 2, we first prove that the set $\mathcal{S}_j(u)$ (cf. (27)) consists of at most isolated points. Indeed, on the contrary, suppose that there exists a sequence $x_i \in \mathcal{S}_j(u) \to x_\star \in \mathcal{S}_j(v)$. Write $u = (V, v)$ near $x_\star$ as in (28). As shown in Section 5, there exists $\epsilon_0 > 0$ such that $Ord^v(x_i) \geq 1 + \epsilon_0$. Let $\sigma_i > 0$ be equal to twice the distance between x_i and $x_\star$. As in Lemma 36, (a subsequence of) the blow up maps $\{v_{\sigma_i}\}$ of v at $x_\star$ with blow up factor $\sqrt{\frac{I^v(\sigma_i)}{\sigma_i^{n-1}}}$ is a sequence of asymptotic harmonic maps and converges locally uniformly in the pullback sense to a homogeneous harmonic map v_0. Let ξ_i be the point corresponding to $\frac{x_i}{\sigma_i}$ after identifying $x_\star = 0$ via normal coordinates centered at $x_\star$. Thus, $\xi_i \in \partial B_{\frac{1}{2}}(0)$ and $Ord^{v_{\sigma_i}}(\xi_i) \geq 1 + \epsilon_0$. By taking a subsequence if necessary, we can assume that $\xi_i \to \xi_\star \in \partial B_{\frac{1}{2}}(0)$. The upper semicontinuity of order (for example, see the proof of [GS] Lemma 6.5) implies $Ord^{v_0}(\xi_\star) \geq 1 + \epsilon_0$. The homogeneity of v_0 implies $Ord^{v_0}(t\xi_\star) \geq 1 + \epsilon_0$ for all $t \in (0, 2)$. This contradicts the fact that the dimension of the domain is 2 and the Hausdorff codimension of the set of higher order points of a harmonic map is at least 2 (cf. Corollary 11). Thus, we have shown that $\mathcal{S}_j(u)$ consists of at most isolated points.

To complete the proof, recall that the singular set $\mathcal{S}(u)$ consists of points of order > 1 and of points in $\mathcal{S}_j(u)$ for some $j \in \{1, \ldots, k-1\}$. In the first case, they must be zeroes of the Hopf differential of u, and hence they are discrete by holomorphicity. In the second case, they are discrete by the claim above. In any case, given $x \in \mathcal{S}(u)$, there is $r > 0$ such

that $\overline{B_r(x)} \cap \mathcal{S}(u) = \{x\}$. Thus $\partial B_r(x) \subset \mathcal{R}(u)$. Applying Lemma 39 for $\Sigma = \partial B_r(x)$, we have that $u(\partial B_r(x)) \subset \mathcal{T}'$ for some stratum $\mathcal{T}'$ of $\overline{\mathcal{T}}$. Now recall the existence of a convex exhaustion function $f : \mathcal{T}' \to [0, \infty)$ (cf. [Wo3]). Since $u(\partial B_r(x))$ is closed, there exists $c > 0$ such that $u(\partial B_r(x)) \subset \{p \in \mathcal{T}' : f(p) \leq c\}$. Since sublevel sets of a convex function are convex, we conclude $u(\overline{B_r(x)}) \subset \{p \in \mathcal{T}' : f(p) \leq c\}$, and hence $x \in \mathcal{R}(u)$. This contradicts the assumption that $x \in \mathcal{S}(u)$ and proves $\mathcal{S}(u) = \emptyset$. Q.E.D.

7 Applications

In this section, we prove our main result Theorem 3 as well as Theorem 4.

PROOF OF THEOREM 3. Equipped with our main technical results, Theorem 1 and Theorem 2, the argument follows along the lines of [JoYa]. First, as in [JoYa], we construct another Kähler manifold Γ-equivariantly biholomorphic to $\tilde{M}$ (which we call again $\tilde{M}$ for the sake of simplicity) and a finite energy Γ-equivariant Lipschitz map $\tilde{f} : \tilde{M} \to \mathcal{T}$. Then [DW] implies that there exists a Γ-equivariant harmonic map $\tilde{u} : \tilde{M} \to \overline{\mathcal{T}}$. By Lemma 39, there exists a stratum $\mathcal{T}'$ of $\overline{\mathcal{T}}$ such that $u(\mathcal{R}(u)) \subset \mathcal{T}'$. Thus, $u(\tilde{M}) \subset \overline{\mathcal{T}}'$, and $\overline{\mathcal{T}}'$ must be invariant by the entire mapping class group $Mod(S)$ by the equivariance of u. This implies $\mathcal{T}' = \mathcal{T}$; in other words,

$$u(\mathcal{R}(u)) \subset \mathcal{T}.$$

As in [GS] or [DMV], the stong negative curvature of $\mathcal{T}$ together with Theorem 1 and Theorem 2 imply that u is pluriharmonic on the regular set $\mathcal{R}(u)$. More precisely, on $\mathcal{R}(u)$, we have that

$$D''d'u \equiv 0 \equiv D'd''u \quad \text{and} \quad \sum_{i,j,k,l} R_{ijkl} d''u_i \wedge d'u_j \wedge d'u_k \wedge d''u_l \equiv 0. \tag{83}$$

Next, by [Schi] Lemma 2, given any $x \in \mathcal{S}(u)$ there exists a holomorphic disc Σ through x such that

$$\mathcal{H}^1(\mathcal{S}(u) \cap \Sigma) = 0. \tag{84}$$

We next need the following

Claim 40 *The restriction of u to Σ is a harmonic map.*

PROOF. Let $w : \Sigma \to \overline{\mathcal{T}}$ be a harmonic map with $w|_{\partial\Sigma} = u|_{\partial\Sigma}$. We will show $u = w$, thereby proving the claim. Fix $\varphi \in C_c^\infty(\Sigma)$ with $0 \leq \varphi \leq 1$. For $\epsilon > 0$, (84) implies that there exists a covering $\{B_{r_i}(x_i)\}_{i=1}^N$ of $\sup(\varphi) \cap \mathcal{S}(u) \subset \Sigma$ such that $\sum_{i=1}^N r_i < \epsilon$. Let ϕ_i be a smooth function such $0 \leq \phi_i \leq 1$, $\phi_i \equiv 0$ in $B_{r_i}(x_i)$, $\phi_i \equiv 1$ outside $B_{2r_i}(x_i)$ and $|\nabla \phi_i| < \frac{1}{r_i}$. Define $\phi_\epsilon = \Pi_{i=1}^N \phi_i$ and $\phi_\epsilon^i = \Pi_{j \neq i} \phi_j$. Since u is pluriharmonic in $\mathcal{R}(u)$, its restriction $u|_{\Sigma \backslash \bigcup_{i=1}^N B_{r_i}(x_i)}$ is a harmonic map. Thus, $d^2(u,w)$ is weakly subharmonic in $\Sigma \backslash \bigcup_{i=1}^N B_{r_i}(x_i)$ (cf. [KS1] Lemma 2.4.2 and Remark 2.4.3). Thus,

$$\begin{aligned}
&\int_\Sigma \phi_\epsilon \nabla \varphi \cdot \nabla d^2(u,w) dxdy + \sum_{i=1}^N \int_{B_{2r_i}(x_i)} \varphi \phi_\epsilon^i \nabla \phi_i \cdot \nabla d^2(u,w) dxdy \\
= \ &\int_\Sigma \nabla(\varphi \phi_\epsilon) \cdot \nabla d^2(u,w) dxdy \geq 0.
\end{aligned}$$

Since $d^2(u,w)$ is a Lipschitz function in $\mathrm{supp}(\varphi)$, we can estimate

$$\sum_{i=1}^N \int_{B_{2r_i}(x_i)} \left| \varphi \phi_\epsilon^i \nabla \phi_i \cdot \nabla d^2(u,w) \right| dxdy \leq C \sum_{i=1}^N r_i^{-1} \int_{B_{2r_i}(x_i)} dxdy \leq C \sum_{i=1}^N r_i < C\epsilon.$$

Letting $\epsilon \to 0$, we obtain

$$\int_\Sigma \nabla \varphi \cdot \nabla d^2(u,w) dxdy \geq 0.$$

In other words, $d^2(u,w)$ is a weakly subharmonic function on Σ. Since u and w agree on the boundary $\partial\Sigma$, we conclude that $u = w$ on Σ. Q.E.D.

We now complete the proof of Theorem 3. Indeed, by Claim 40 and Theorem 5, the restriction $u|_\Sigma$ maps into $\mathcal{T}$. This in turn implies that $\mathcal{S}(u) = \emptyset$, and hence u maps into $\mathcal{T}$. The assertion of the theorem now follows by [JoYa] *c*), *d*) and *e*). Q.E.D.

PROOF OF THEOREM 4. As in [GS] Lemma 8.1, we first construct a finite energy equivariant Lipschitz map $f : \tilde{M} = G/K \to \mathcal{T}$. Under the assumption that $\rho : \Lambda \to Mod(S)$ is sufficiently large, [DW] implies that there exists a Λ-equivariant harmonic map

$$u : \tilde{M} \to \overline{\mathcal{T}}.$$

By Lemma 39, there exists $\mathcal{T}' \subset \overline{\mathcal{T}}$ such that $u(\mathcal{R}(u)) \subset \mathcal{T}'$. We are going to show that u is constant, so with an intent of arriving at a contradiction, let's assume that u is non-constant. As in [DMV] Corollary 14 and Lemma 15, our regularity Theorem 1 and Theorem 2 imply that u is totally geodesic on the regular set $\mathcal{R}(u)$. In other words, u satisfies on $\mathcal{R}(u)$

$$\nabla du = 0. \tag{85}$$

As in [DMV] proof of Theorem 1, (85) combined with Theorem 1 implies that u is totally geodesic on the entire $\tilde{M}$ in the sense that u maps geodesics to geodesics.

Since the domain is an irreducible symmetric space, u must be a totally geodesic *immersion* into a Teichmüller space $\mathcal{T}'$. This is clearly a contradiction if the symmetric space has rank ≥ 2. In the rank 1 case, the contradiction follows from [Wu] Theorem 1.2. We thus conclude that u is constant, hence $\rho(\Lambda)$ fixes a point in Teichmüller space. Since the action of the mapping class group is properly discontinuous, this implies that $\rho(\Lambda)$ is finite. Q.E.D.

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